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PART IV.

HYDRAULICS.

CHAPTER I.

DEFINITIONS—FLUID PRESSURE—HYDROSTATICS BEGUN.

406. A Perfect Fluid is a substance the particles of which are capable of moving upon each other with the greatest freedom, absolutely without friction, and are destitute of mutual attraction. In other words, the stress between any two contiguous portions of a perfect fluid is always one of *compression* and *normal* to the dividing surface at every point; i.e., *no shear* or tangential action can exist on any imaginary cutting plane.

Hence if a perfect fluid is contained in a vessel of rigid material the pressure experienced by the walls of the vessel is *normal to the surface of contact at all points.*

For the practical purposes of Engineering, water, alcohol, mercury, air, steam, and all gases may be treated as perfect fluids within certain limits of temperature.

407. Liquids and Gases.—A fluid a definite mass of which occupies a definite volume at a given temperature, and is incapable both of expanding into a larger volume and of being compressed into a smaller volume at that temperature, is called a **Liquid**, of which water, mercury, etc., are common examples; whereas a **Gas** is a fluid a mass of which is capable of almost indefinite expansion or compression, according as the space within the confining vessel is made larger or smaller, and always tends to fill the vessel, which must therefore be closed in every direction to prevent its escape.

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Liquids are sometimes called *inelastic* fluids, and gases *elastic* fluids.

408. Remarks.—Though practically we may treat all liquids as incompressible, experiment shows them to be compressible to a slight extent. Thus, a cubic inch of water under a pressure of 15 lbs. on each of its six faces loses only fifty millionths (0.000050) of its original volume, while remaining at the same temperature; if the temperature be sufficiently raised, however, its bulk will remain unchanged (provided the initial temperature is over 40° Fahr.). Conversely, by heating a liquid in a rigid vessel completely filled by it, a great bursting pressure may be produced. The slight cohesion existing between the particles of most liquids is too insignificant to be considered in the present connection.

The property of indefinite expansion, on the part of gases, by which a confined mass of gas can continue to fill a confined space which is progressively enlarging, and exert pressure against its walls, is satisfactorily explained by the "Kinetic Theory of Gases," according to which the gaseous particles are perfectly elastic and in continual motion, impinging against each other and the confining walls. Nevertheless, for practical purposes, we may consider a gas as a continuous substance.

Although by the abstraction of heat, or the application of great pressure, or both, all known gases may be reduced to liquids (some being even solidified); and although by converse processes (imparting heat and diminishing the pressure) liquids may be transformed into gases, the range of temperature and pressure in all problems to be considered in this work is supposed kept within such limits that no extreme changes of state, of this character, take place. A gas approaching the point of liquefaction is called a **Vapor.**

Between the solid and the liquid state we find all grades of intermediate conditions of matter. For example, some substances are described as soft and plastic solids, as soft putty, moist earth, pitch, fresh mortar, etc.; and others as viscous and sluggish liquids, as molasses and glycerine. In sufficient bulk, however, the latter may still be considered as perfect fluids. Even water is slightly viscous.

409. Heaviness of Fluids.—The weight of a cubic unit of a homogeneous fluid will be called its *heaviness*, or rate of weight (see § 7), and is a measure of its density. Denoting it by γ, and the volume of a definite portion of the fluid by V, we have, for the weight of that portion,

$$G = V\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

This, like the great majority of equations used or derived in this work, is of *homogeneous form* (§ 6), i.e., admits of any system of units. E.g., in the metre-kilogram-second system, if γ is given in kilos. per cubic metre, V must be expressed in cubic metres, and G will be obtained in kilos.; and similarly in any other system. The quality of γ, $= G \div V$, is evidently one dimension of force divided by three dimensions of length.

In the following table, in the case of gases, the temperature and pressure are mentioned at which they have the given heaviness, since under other conditions the heaviness would be different; in the case of liquids, however, for ordinary purposes the effect of a change of temperature may be neglected (within certain limits).

HEAVINESS OF VARIOUS FLUIDS.*

[In ft. lb. sec. system; γ = weight in lbs. of a cubic foot.]

Liquids.		Gases { At temp. of melting ice; and 14.7 lbs. per sq. in. tension.	
Fresh water, $\gamma =$	62.5	Atmospheric Air	0.08076
Sea water	64.0	Oxygen	0.0892
Mercury	848.7	Nitrogen	0.0786
Alcohol	49.3	Hydrogen	0.0056
Crude Petroleum, about	55.0	Illuminating Gas, } from	0.0300
(N.B.—A cubic inch of water weighs 0.036024 lbs.; and a cubic foot 1000 av. oz.)		} to	0.0400
		Natural Gas, about	0.0500

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* See Trautwine's Civ. Engineer's Pocket Book for an extended table—p. 380, edition of 1885.

For use in problems where needed, values for the heaviness of pure fresh water are given in the following table (from Rossetti) for temperatures ranging from freezing to boiling; as also the relative density, that at the temperature of maximum density, 39°.3 Fahr. being taken as unity. The temperatures are Fahr., and γ is in lbs. per cubic foot.

Temp.	Rel. Dens.	γ.	Temp.	Rel. Dens.	γ.	Temp.	Rel. Dens.	γ.
32°	.99987	62.416	60°	.99907	62.366	140°	.98338	61.386
35°	.99996	62.421	70°	.99802	62.300	150°	.98043	61.203
39°.3	1.00000	62.424	80°	.99669	62.217	160°	.97729	61.006
40°	.99999	62.423	90°	.99510	62.118	170°	.97397	60.799
43°	.99997	62.422	100°	.99318	61.998	180°	.97056	60.586
45°	.99992	62.419	110°	.99105	61.865	190°	.96701	60.365
50°	.99975	62.408	120°	.98870	61.719	200°	.96333	60.135
55°	.99946	62.390	130°	.98608	61.555	212°	.95865	59.843

From D. K. Clark's "Manual."	for temp. =	230°	250°	270°	290°	298°	338°	366°	390°
	γ =	59.4	58.7	58.2	57.6	57.3	56.1	55.3	54.5

EXAMPLE 1. What is the heaviness of a gas, 432 cub. in. of which weigh 0.368 ounces? Use ft.-lb.-sec. system.

432 cub. in. $= \frac{1}{4}$ cub. ft. and 0.368 oz. $= 0.023$ lbs.

$$\therefore \gamma = \frac{G}{V} = \frac{0.023}{\frac{1}{4}} = 0.092 \text{ lbs. per cub. foot.}$$

EXAMPLE 2. Required the weight of a right prism of mercury of 1 sq. inch section and 30 inches altitude.

$V = 30 \times 1 = 30$ cub. in. $= \frac{30}{1728}$ cub. feet; while from the table, γ for mercury $= 848.7$ lbs. per cub. ft.

$$\therefore \text{its weight} = G = V\gamma = \frac{30}{1728} \times 848.7 = 14.73 \text{ lbs.}$$

410. Definitions.—By *Hydraulics* we understand the mechanics of fluids as utilized in Engineering. It may be divided into

Hydrostatics, treating of fluids at rest; and

Hydrodynamics (or *Hydrokinetics*), which deals with fluids in motion. (The name *Pneumatics* is sometimes used to cover both the statics and dynamics of gaseous fluids.)

[Rankine's nomenclature has been adopted in the present work. Some recent writers use the term *Hydromechanics* for mechanics of fluids, subdividing it into *Hydrostatics* and *Hydrokinetics*, as above; they also use the term Dynamics to embrace both of the two divisions called Statics and Dynamics by Rankine, which by them are called Statics and Kinetics respectively. Though unusual, perhaps, the term Hydraulics is here used to cover the applied Mechanics of Gases as well as of Liquids.]

Before treating separately of liquids and gases, a few paragraphs will be presented applicable to both kinds of fluids.

411. Pressure per Unit Area, or Intensity of Pressure.—As in § 180 in dealing with solids, so here with fluids we indicate the pressure per unit area between two contiguous portions of fluid, or between a fluid and the wall of the containing vessel, by p, so that if dP is the total pressure on a small area dF, we have

$$p = \frac{dP}{dF} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as the pressure per unit area, or intensity of pressure (often, though ambiguously, called the *tension* in speaking of a gas) on the small surface dF. If pressure of the *same intensity* exists over a finite plane surface of area $= F$, the total pressure on that surface is

$$\left.\begin{array}{r} P = \int p\,dF = p\int dF = Fp, \\ \text{or} \qquad p = \dfrac{P}{F}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

(N.B.—For brevity the single word "pressure" will sometimes be used, instead of intensity of pressure, where no ambiguity can arise.) Thus, it is found that, under ordinary conditions at the sea level, the atmosphere exerts a normal pressure (normal, because fluid pressure) on all surfaces, of an intensity of about $p = 14.7$ lbs. per sq. inch ($= 2116.$ lbs. per sq. ft.). This intensity of pressure is called *one atmosphere*. For example, the total atmospheric pressure on a surface of 100 sq. in. is [inch, lb., sec.]

$$P = Fp = 100 \times 14.7 = 1470 \text{ lbs.} \quad (= 0.735 \text{ tons.})$$

The quality of p is evidently one dimension of force divided by two dimensions of length.

412. Hydrostatic Pressure; per Unit Area, in the Interior of a Fluid at Rest.—In a body of fluid of uniform heaviness, at rest, it is required to find the mutual pressure per unit area between the portions of fluid on opposite sides of any imaginary cutting plane. As customary, we shall consider portions of the fluid as free bodies, by supplying the forces exerted on them by all contiguous portions (of fluid or vessel wall), also those of the earth (their weights), and then apply the conditions of equilibrium.

First, cutting plane horizontal.—Fig. 451 shows a body of homogeneous fluid confined in a rigid vessel closed at the top with a small air-tight but frictionless piston (a horizontal disk) of weight $= G$ and exposed to atmospheric pressure ($= p_a$ per unit area) on its upper face. Let the area of piston-face be $= F$. Then for the equilibrium of the piston the total pressure between its under surface and the fluid at O must be

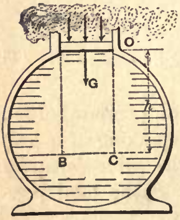

FIG. 451.

$$P = G + Fp_a,$$

and hence the intensity of this pressure is

$$p_0 = \frac{G}{F} + p_a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is now required to find the intensity, p, of fluid pressure between the portions of fluid contiguous to the horizontal cutting plane BC at a vertical distance $= h$ vertically below the piston O. In Fig. 452 we have as a free body the right parallelopiped *OBC* of Fig. 451 with vertical sides (two || to paper and four ⅂ to it). The pressures acting on its six faces are normal to them respectively, and the weight of the prism is = vol. $\times \gamma = Fh\gamma$, supposing γ to have the same value at all parts of the column (which is practically true for any height of liquid and for a small height of gas). Since the prism is in equilibrium under the forces shown in the figure, and would still be so were it to become rigid, we may put (§ 36) Σ (vert. compons.) = 0 and hence obtain

$$Fp - Fp_0 - Fh\gamma = 0. \quad . \quad . \quad (2)$$

(In the figure the pressures on the vertical faces || to paper have no vertical components, and hence are not drawn.) From (2) we have

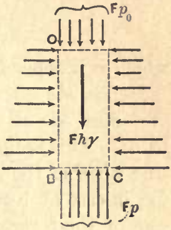

FIG. 452.

$$p = p_0 + h\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

($h\gamma$, being the weight of a column of homogeneous fluid of unity cross-section and height h, would be the total pressure on the base of such a column, if at rest and with no pressure on the upper base, and hence might be called *intensity due to weight.*)

Secondly, cutting plane oblique.—Fig. 453. Consider free an infinitely small right triangular prism *bcd*, whose bases are || to the paper, while the three side faces (rectangles), having areas = dF, dF_1, and dF_2, are respectively horizontal, vertical, and oblique; let angle $cbd = \alpha$. The surface *bc* is a portion of the plane *BC* of Fig. 452. Given p (= intensity of pressure on dF) and α, required p_2, the intensity of pressure on the oblique face *bd*, of area dF_2. [N. B.—The prism is taken very small in order that the intensity of pressure may be considered constant over any one face; and also that the weight of the prism may be neglected, since it involves the volume (three dimen-

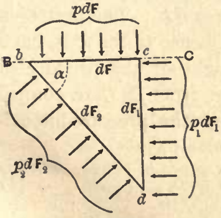

FIG. 453.

sions) of the prism, while the total face pressures involve only two, and is hence a differential of a higher order.]

From Σ (*vert. compons.*) = 0 we shall have

$$p_2 dF_2 \cos \alpha - pdF = 0 ; \text{ but } dF \div dF_2 = \cos \alpha ;$$

$$\therefore \quad p_2 = p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is independent of the angle α.

Hence, *the intensity of fluid pressure at a given point is the same on all imaginary cutting planes containing the point.* This is the most important property of a fluid, and is true whether the liquid is at rest or has *any kind of motion ;* for, in case of rectilinear accelerated motion, *e.g.*, although the sum of the force-components in the direction of the acceleration does not in general = 0, but = mass × acc., still, the mass of the body in question is = *weight* ÷ g, and therefore the term mass × acc. is a differential of a higher order than the other terms of the equation, and hence the same result follows as when there is no motion (or uniform rectilinear motion).

413. The Intensity of Pressure is Equal at all Points of any Horizontal Plane in a body of homogeneous fluid at rest. If we consider a right prism of the fluid in Fig. 451, of small vertical thickness, its axis lying in any horizontal plane BC, its bases will be vertical and of equal area dF. The pressures on its sides, being normal to them, and hence to the axis, have no components ‖ to the axis. The weight of the prism also has no horizontal component. Hence from Σ (hor. comps. ‖ to axis) = 0, we have, p_1 and p_3 being the pressure-intensities at the two bases,

$$p_1 dF - p_3 dF = 0 ; \quad \therefore p = p_3 , \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which proves the statement at the head of this article.

It is now plain, from this and the preceding article, that the pressure-intensity p at any point in a homogeneous fluid *at rest* is *equal to that at any higher point, plus the weight*

($h\gamma$) of a column of the fluid of section unity and of altitude (h) = vertical distance between the points.

i.e., $$p = p_0 + h\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whether they are in the same vertical or not, and whatever be the shape of the containing vessel (or pipes), provided the fluid is continuous between the two points; for, Fig. 454, by considering a series of small prisms, alternately vertical and horizontal, *obcde*, we know that

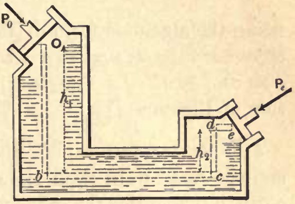

FIG. 454.

$$p_b = p_0 + h_1\gamma; \qquad p_c = p_b;$$
$$p_d = p_c - h_2\gamma; \text{ and } p_e = p_d;$$

hence, finally, by addition we have

$$p_e = p_0 + h\gamma$$

(in which $h = h_1 - h_2$).

If, therefore, upon a small piston at o, of area $= F_0$, a force P_0 be exerted, and an inelastic fluid (liquid) completely fills the vessel, then, for equilibrium, the force to be exerted upon the piston at e, viz., P_e, is thus computed: For equilibrium of fluid $p_e = p_0 + h\gamma$; and for equil. of piston o, $p_0 = P_0 \div F_0$; also, $p_e = P_e \div F_e$;

$$\therefore P_e = \frac{F_e}{F_0} P_0 + F_e h\gamma. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (3) we learn that if the pistons are at the same level ($h = 0$) the total pressures on their inner faces are *directly proportional to their areas.*

If the fluid is gaseous (2) and (3) are practically correct if h is not $>$ 100 feet (for, gas being compressible, the lower strata are generally more dense than the upper), but in (3) the pistons must be fixed, and P_e and P_0 refer solely to the interior pressures.

Again, if h is small or p_0 very great, the term $h\gamma$ may be omitted altogether in eqs. (2) and (3) (especially with gases, since for them γ (heaviness) is usually small), and we then have, from (2),

$$p = p_0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

being the algebraic form of the statement: *A body of fluid at rest transmits pressure with equal intensity in every direction and to all of its parts.* [Principle of "Equal Transmission of Pressure."]

414. Moving Pistons.—If the fluid in Fig. 454 is *inelastic* and the vessel walls rigid, the motion of one piston (o) through a distance s_0 causes the other to move through a distance s_e determined by the relation $F_0 s_0 = F_e s_e$ (since the volumes described by them must be equal, as liquids are incompressible); but on account of the inertia of the liquid, and friction on the vessel walls, equations (2) and (3) no longer hold exactly, still are approximately true if the motion is very slow and the vessel short, as with the cylinder of a water-pressure engine.

But if the fluid is compressible and elastic (gases and vapors; steam, or air) and hence of small density, the effect of inertia and friction is not appreciable in short wide vessels like the cylinders of steam- and air-engines, and those of air-compressors; and eqs. (2) and (3) still hold, practically, even with high piston-speeds. For example, in the space AB, Fig. 455, between the piston and cylinder-head of a steam-engine (piston moving toward the right) the intensity of pressure, p, of the steam against the moving piston B is practically equal to that against the cylinder-head A at the same instant.

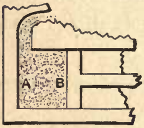

FIG. 455.

415. An Important Distinction between gases and liquids (i.e., between elastic and inelastic fluids) consists in this:

A liquid can exert pressure against the walls of the containing vessel only by its weight, or (when confined on all sides) by transmitted pressure coming from without (due to piston pressure, atmospheric pressure, etc.); whereas—

A gas, confined, as it must be, on all sides to prevent diffusion, exerts pressure on the vessel not only by its weight, but by its elasticity or tendency to expand. If pressure from without is also applied, the gas is compressed and exerts a still greater pressure on the vessel walls.

416. Component, of Pressure, in a Given Direction.—Let $ABCD$, whose area $= dF$, be a small element of a surface, plane or curved, and p the intensity of fluid pressure upon this element, then the total pressure upon it is pdF, and is of course normal to it. Let $A'B'CD$ be the projection of the element dF upon a plane CDM making an angle α with the element, and let it be required to find the value of the component of pdF in a direction normal to this last plane (the other component being understood to be || to the same plane). We shall have

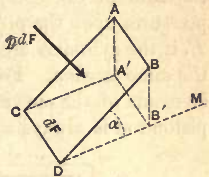

FIG. 455.

$$\textit{Compon. of } pdF \urcorner \textit{ to } CDM = pdF\cos\alpha = p(dF.\cos\alpha). \quad (1)$$

But $dF \,.\, \cos\alpha =$ area $A'B'CD$, the projection of dF upon the plane CDM.

$$\therefore \textit{ Compon.} \urcorner \textit{ to plane } CDM = p \times (\textit{project. of } dF \textit{ on } CDM);$$

i.e., *the component of fluid pressure* (on an element of a surface) *in a given direction* (the other component being ⌉ to the first) *is found by multiplying the intensity of the pressure by the area of the projection of the element upon a plane* ⌉ *to the given direction.*

It is seen, as an example of this, that if the fluid pressures on the elements of the inner surface of one hemisphere of a hollow sphere containing a gas are resolved into components ⌉ and || to the plane of the circular base of the hemisphere, the sum of the former components simply $= \pi r^2 p$, where r is the radius of the sphere, and p the intensity of the fluid pressure; for, from the foregoing, the sum of these components is just the same as the total pressure would be, having an intensity p, on a great circle of the sphere, the area, πr^2, of this circle being the sum of the areas of the projections, upon this circle as a base, of all the elements of the hemispherical surface. (Weight of fluid neglected.)

A similar statement may be made as to the pressures on the inner curved surface of a right cylinder.

417. Non-planar Pistons.—From the foregoing it follows that the sum of the components || to the piston-rod, of the fluid pressures upon the piston at A, Fig. 457, is just the same as at B, if the cylinders are of equal size and the steam, or air, is at the same tension. For the sum of the projections of all the elements of the curved surface of A upon a plane ⏋ to the piston-rod is always $= \pi r^2 =$ area of section of cylinder-bore.

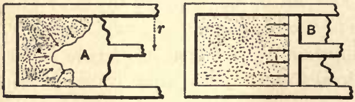

Fig. 457.

If the surface of A is symmetrical about the axis of the cylinder the other components (i.e., those ⏋ to the piston-rod) will neutralize each other. If the line of intersection of that surface with the surface of the cylinder is not symmetrical about the axis of the cylinder, the piston may be pressed laterally against the cylinder-wall, but the thrust along the rod or "*working force*" (§ 128) is the same (except for friction induced by the lateral pressure), in all instances, as if the surface were plane and ⏋ to piston-rod.

418. Bramah, or Hydraulic, Press.—This is a familiar instance of the principle of transmission of fluid pressure. Fig. 458. Let the small piston at O have a diameter $d = 1$ inch $= \frac{1}{12}$ ft., while the plunger E, or large piston, has a diameter $d' = AB = CD = 15$ in. $= \frac{5}{4}$ ft. The lever MN weighs $G_1 = 3$ lbs., and a weight $G = 40$ lbs. is hung at M. The lever-arms of these forces about the fulcrum N are given in the figure. The apparatus being full of water (oil is often used), the fluid pressure P_0 against the small piston is found by putting

Σ(moms. about N) $= 0$ for the equilibrium of the lever; whence [ft., lb., sec.]

$$P_0 \times 1 - 40 \times 3 - 3 \times 2 = 0. \quad \therefore P_0 = 126 \text{ lbs.}$$

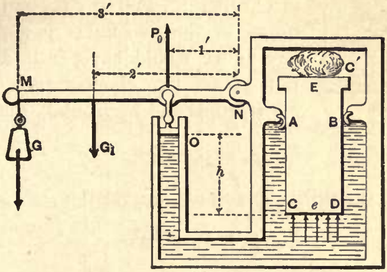

FIG. 458.

But, denoting atmospheric pressure by p_a, and that of the water against the piston by p_0 (per unit area), we may also write

$$P_0 = F_0 p_0 - F_0 p_a = \tfrac{1}{4}\pi d^2 (p_0 - p_a).$$

Solving for p_0, we have, putting $p_a = 14.7 \times 144$ lbs. per sq. ft.,

$$p_0 = \left[126 \div \frac{\pi}{4}\left(\tfrac{1}{12}\right)^2\right] + 14.7 \times 144 = 25236 \text{ lbs. per sq. ft.}$$

Hence at e the press. per unit area, from § 409, and (2), § 413, is

$$p_e = p_0 + h\gamma = 25236 + 3 \times 62.5 = 25423 \text{ lbs. per sq. ft.}$$

$=$ 175.6 lbs. per sq. inch or 11.9 atmospheres, and the total upward pressure at e on base of plunger is

$$P = F_e p_e = \pi \frac{d'^2}{4} p_e = \tfrac{1}{4}\pi\left(\tfrac{5}{4}\right)^2 \times 25423 = 31194 \text{ lbs.},$$

or almost 16 tons (of 2000 lbs. each). The compressive force upon the block or bale, C, $= P$ less the weight of the plunger and total atmos. pressure on a circle of 15 in. diameter.

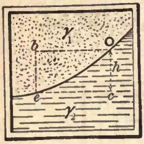

FIG. 459.

419. The Dividing Surface of Two Fluids (which do not mix) in Contact, and at Rest, is a Horizontal Plane.—For, Fig. 459, supposing any two points e and O of this surface to be at different levels (the pressure at O being p_0, that at e p_e, and the heavinesses of the two fluids γ_1 and γ_2 respectively), we would have, from a consideration of the two elementary prisms eb and bO (vertical and horizontal), the relation

$$p_e = p_0 + h\gamma_1;$$

while from the prisms ec and cO, the relation

$$p_e = p_0 + h\gamma_2.$$

These equations are conflicting, hence the above supposition is absurd. Therefore the proposition is true.

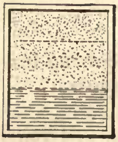

FIG. 460.

For stable equilibrium, evidently, the heavier fluid must occupy the lowest position in the vessel, and if there are several fluids (which do not mix), they will arrange themselves vertically, in the order of their densities, the heaviest at the bottom, Fig. 460. On account of the property called *diffusion* the particles of two gases placed in contact soon intermingle and form a uniform mixture. This fact gives strong support to the "Kinetic Theory of Gases" (§ 408).

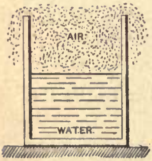

FIG. 461.

420. Free Surface of a Liquid at Rest.—The surface (of a liquid) not in contact with the walls of the containing vessel is called a *free surface*, and is necessarily horizontal (from § 419) when the liquid is at rest. Fig. 461. (A gas, from its tendency to indefinite expansion, is incapable of having a free surface.) This is true even if the space above the liquid is vacuous, for if the surface were inclined or curved, points in the body of the liquid and in the same horizontal plane would have different heights (or "heads") of liquid between them and the surface, producing different intensities of pressure in the plane, which is contrary to § 413.

When large bodies of liquid like the ocean are considered, gravity can no longer be regarded as acting in parallel lines; consequently the free surface of the liquid is curved, being ⏋ to the direction of (apparent) gravity at all points. For ordinary engineering purposes (except in Geodesy) the free surface of water at rest is a horizontal plane.

421. Two Liquids (which do not mix) at Rest in a Bent Tube open at Both Ends to the Air, Fig. 460; water and mercury, for instance. Let their heavinesses be γ_1 and γ_2 respectively. The pressure at e may be written (§ 413) either

$$p_e = p_{0_1} + h_1\gamma_1 \quad . \quad . \quad . \quad (1)$$

or

$$p_e = p_{0_2} + h_2\gamma_2 \quad . \quad . \quad . \quad (2)$$

according as we refer it to the water column or the mercury column and their respective free surfaces where the pressure $p_{0_1} = p_{0_2} = p_a =$ atmos. press.

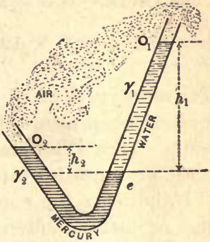

FIG. 462.

e is the surface of contact of the two liquids. Hence we have

$$p_a + h_1\gamma_1 = p_a + h_2\gamma_2; \quad \text{i.e., } h_1 : h_2 :: \gamma_2 : \gamma_1. \quad . \quad (3)$$

i.e., *the heights of the free surfaces of the two liquids above the surface of contact are inversely proportional to their respective heavinesses.*

EXAMPLE.—If the pressure at $e = 2$ atmospheres (§ 396) we shall have from (1) (inch-lb.-sec. system of units)

$$h_2\gamma_2 = p_e - p_a = 2 \times 14.7 - 14.7 = 14.7 \text{ lbs. per sq. inch.}$$

$$\therefore h_2 \text{ must} = 14.7 \div [848.7 \div 1728] = 30 \text{ inches}$$

(since, for mercury, $\gamma_2 = 848.7$ lbs. per cub. ft.). Hence, from (3),

$$h_1 = \frac{h_2\gamma_2}{\gamma_1} = \frac{30 \times [848.7 \div 1728]}{62.5 \div 1728} = 408 \text{ inches} = 34 \text{ feet.}$$

i.e., for equilibrium, and that p_e may $=$ 2 atmospheres, h_2 and h_1 (of mercury and water) must be 30 in. and 34 feet respectively.

422. City Water-pipes.—If $h =$ vertical distance of a point B of a water-pipe below the free surface of reservoir, and the water be at rest, the pressure on the inner surface of the pipe at B (per unit of area) is

$$p = p_0 + h\gamma\,; \quad \text{and here } p_0 = p_a = \text{atmos. press.}$$

EXAMPLE.—If $h =$ 200 ft. (using the inch, lb., and second)

$$p = 14.7 + [200 \times 12][62.5 \div 1728] = 101.5 \text{ lbs. per sq. in.}$$

The term $h\gamma$, alone, $=$ 86.8 lbs. per sq. inch, is spoken of as the *hydrostatic pressure* due to 200 feet height, or "**Head**," of water. (See Trautwine's Pocket Book for a table of hydrostatic pressures for various depths.)

If, however, the water is *flowing* through the pipe, the pressure against the interior wall becomes less (a problem of Hydrodynamics to be treated subsequently), while if that motion is suddenly checked, the pressure becomes momentarily much greater than the hydrostatic. This shock is called "water-ram" and "water-hammer," and may be as great as 200 to 300 lbs. per sq. inch.

423. Barometers and Manometers for Fluid Pressure.—If a tube, closed at one end, is filled with water, and the other extremity is temporarily stopped and afterwards opened under water, the closed end being then a (vertical) height $= h$ above the surface of the water, it is required to find the intensity, p_0, of fluid pressure at the top of the tube, supposing it to remain filled with water. Fig. 463. At E inside the tube the pressure is 14.7 lbs. per sq. inch, the same as that outside at the same level (§ 413); hence, from $p_E = p_0 + h\gamma$,

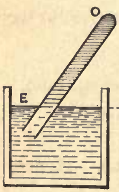

FIG. 463.

$$p_0 = p_E - h\gamma \,. \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)'$$

EXAMPLE.—Let $h = 10$ feet (with inch-lb.-sec. system); then

$$p_0 = 14.7 - 120 \times [62.5 \div 1728] = 10.4 \text{ lbs. per sq. inch,}$$

or about $\frac{2}{3}$ of an atmosphere. If now we inquire the value of h to make $p_0 =$ zero, we put $p_E - h\gamma = 0$ and obtain $h =$ 408 inches, $= 34$ ft., which is called *the height of the water-barometer*. Hence, Fig. 463*a*, ordinary atmospheric pressure will not sustain a column of water higher than 34 feet. If mercury is used instead of water the height supported by one atmosphere is

$$b = 14.7 \div [848.7 \div 1728] = 30 \text{ inches,}$$

$= 76$ centims. (about), and the tube is of more manageable proportions than with water, aside from the advantage that no vapor of mercury forms above the liquid at ordinary temperatures. [In fact, the water-barometer height $b = 34$ feet has only a theoretical existence since at ordinary temperatures (40° to 80° Fahr.) vapor of water would form above the column and depress it by from 0.30 to 1.09 ft.] Such an apparatus is called a *Barometer*, and is used not only for measuring the varying tension of the atmosphere (from 14.5 to 15 lbs. per sq. inch, according to the weather and height above sea-level), but also that of any body of gas. Thus, Fig. 464, the gas in D is put in communication with the space above the mercury in the cistern at C; and we have $p = h\gamma$, where $\gamma =$ heav. of mercury, and p is the pressure on the liquid in the cistern. For delicate measurements an attached thermometer is also used, as the heaviness γ varies slightly with the temperature.

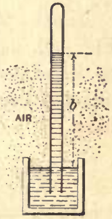

FIG. 463*a*.

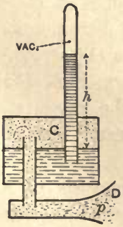

FIG. 464.

If the vertical distance CD is small, the tension in C is considered the same as in D.

For gas-tensions greater than one atmosphere, the tube may be left open at the top, forming an *open ma-*

nometer, Fig. 465. In this case, the tension of the gas above the mercury in the cistern is

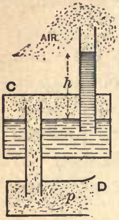

FIG. 465.

$$p = (h + b)\gamma, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which b is the height of mercury (about 30 in.) to which the tension of the atmosphere above the mercury column is equivalent.

EXAMPLE.—If $h = 51$ inches, Fig. 465, we have (ft., lb., sec.)

$$p = [4.25 \text{ ft.} + 2.5 \text{ ft.}]\ 848.7 = 5728 \text{ lbs. per sq. foot}$$

$$= 39.7 \text{ lbs. per sq. inch} = 2.7 \text{ atmospheres.}$$

Another form of the open manometer consists of a U tube, Fig. 464, the atmosphere having access to one branch, the gas to be examined, to the other, while the mercury lies in the curve. As before, we have

$$p = (h + b)\gamma = h\gamma + p_a, \quad . \quad (2)$$

where p_a = atmos. tension, and b as above. The tension of a gas is sometimes spoken of as measured by so many *inches of mercury*. For example, a tension of 22.05 lbs. per sq. inch ($1\frac{1}{2}$ atmos.) is measured by 45 inches of mercury in a vacuum manometer (i.e., a common barometer), Fig. 464. With the open manometer this tension ($1\frac{1}{2}$ atmos.) would be indicated by 15 inches of actual mercury, Figs. 465 and 466. An ordinary steam-gauge indicates the *excess* of tension over one atmosphere; thus "40 lbs. of steam" implies a tension of $40 + 14.7 = 54.7$ lbs. per sq. in.

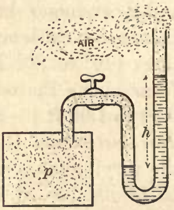

FIG. 466.

The *Bourdon* steam-gauge in common use consists of a curved elastic metal tube of flattened or elliptical section (with the long axis ┐ to the plane of the tube), and has one end fixed. The movement of the other end, which is free and closed, by proper mechanical connection gives motion to the pointer of a dial. This movement is caused by any change of tension in the steam or gas admitted, through the fixed end, to the interior of the tube. As the tension increases the elliptical section becomes less flat, i.e., more nearly circular, causing the two ends of the tube to separate more widely, i.e., the free end moves away from the fixed end; and *vice versâ*.

Such gauges, however, are not always reliable. They are graduated by comparison with mercury manometers; and should be tested from time to time in the same way.*

424. Tension of Illuminating Gas.—This is often spoken of as measured by *inches of water* (from 1 to 3 inches usually). Strictly it should be stated that this water-height measures the *excess* of its tension over that of the atmosphere. Thus, in Fig. 466, water being used instead of mercury, $h =$ say 2 inches, while $b = 408$ inches.

This *difference* of tension may be largely affected by a change in the barometer due to the weather, or by a difference in altitude, as the following example will illustrate:

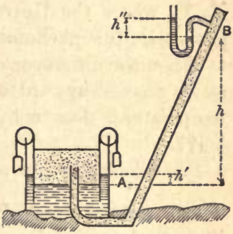

FIG. 467.

EXAMPLE.—Supposing the gas at rest, and the tension at the gasometer A, Fig. 467, to be "two inches of water," required the water-column h'' (in open tube) that the gas will support in the pipe at B, 120 feet (vertically) above the gasometer. Let the temperature be freezing (nearly), and the outside air at a tension of 14.7 lbs. per sq. inch; the heaviness of the gas at this temperature being 0.036 lbs. per cubic foot. For the small difference of 120 ft. we may treat both the atmosphere and the gas as liquids, that is, of constant density throughout the vertical column, and therefore apply the principles of § 413; with the following result:

The tension of the outside air at B, supposed to be at the same temperature as at A, will sustain a water-column less than the 408 inches at A by an amount corresponding to the

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* Of late years gauges have come into use constructed of boxes with corrugated sides of thin metal like the aneroid barometer. Motion of the sides, under varying internal fluid pressure, causes movement of a pointer on a dial.

120 feet of air between, of the heaviness .0807 lbs. per cub. ft. 120 feet of air weighing .0807 lbs. per cub. ft. will balance 0.154 ft. of water weighing 62.5 lbs. per cubic ft., i.e., 1.85 inches of water. Now the tension of the gas at B is also less than its tension at A, but the difference is not so great as with the outside air, for the 120 ft. of gas is lighter than the 120 ft. of air. Since 120 ft. of gas weighing 0.036 lbs. per cubic ft. will balance 0.0691 ft., or 0.83 inches, of water, therefore the difference between the tensions of the two fluids at B is greater than at A by (1.85 − 0.83 =) 1.02 inches; or, at B the total difference is 2.00 + 1.02 = 3.02 inches.

Hence if a small aperture is made in the pipe at B the gas will flow out with greater velocity than at A. At Ithaca, N. Y., where the University buildings are 400 ft. above the gas-works, this phenomenon is very marked.

When the difference of level is great the decrease of tension as we proceed upward in the atmosphere, even with constant temperature, does not follow the simple law of § 413; see § 477.

For velocity of flow of gases through orifices, see § 548, etc.

425. Safety-valves.—Fig. 468. Required the proper weight G to be hung at the extremity of the horizontal lever AB, with fulcrum at B, that the flat disk-valve E shall not be forced upward by the steam pressure, p', until the latter reaches a given value $= p$. Let the weight of the arm be G_1, its centre of gravity being at C, a distance $= c$ from B; the other horizontal distances are marked in the figure.

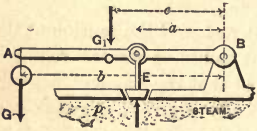

FIG. 468.

Suppose the valve on the point of rising; then the forces acting on the lever are the fulcrum-reaction at B, the weights G and G_1, and the two fluid-pressures on the disk, viz.: Fp_a (atmospheric) downward, and Fp (steam) upward. Hence, from $\Sigma(\text{moms.}_B) = 0$,

$$Gb + G_1c + Fp_a a - Fpa = 0. \quad . \quad . \quad . \quad (1)$$

Solving, we have

$$G = \frac{a}{b}F(p - p_a) - G_1\frac{c}{b}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Example.—With $a = 2$ inches, $b = 2$ feet, $c = 1$ foot $G_1 = 4$ lbs., $p = 6$ atmos., and diam. of disk $= 1$ inch; with the foot and pound,

$$G = \frac{2}{24}\cdot\frac{\pi}{4}\left(\frac{1}{12}\right)^2[6 \times 14.7 \times 144 - 1 \times 14.7 \times 144] - 4 \times \tfrac{1}{2}.$$

$$\therefore G = 2.81 \text{ lbs.}$$

[Notice the cancelling of the 144; for $F(p - p_a)$ is *pounds*, being one dimension of force, if the pound is selected as the unit of force, whether the inch or foot is used in both factors.] Hence when the steam pressure has risen to 6 atmos. (= 88.2 lbs. per square inch) (corresponding to 73.5 lbs. per sq. in. by steam-gauge) the valve will open if $G = 2.81$ lbs., or be on the point of opening.

426. Proper Thickness of Thin Hollow Cylinders (i.e., Pipes and Tubes) to Resist Bursting by Fluid Pressure.

Case I. *Stresses in the cross-section due to End Pressure;* Fig. 469.—Let AB be the circular cap closing the end of a cylindrical tube containing fluid at a tension $= p$. Let $r =$ internal radius of the tube or pipe. Then considering the cap free, neglecting its weight, we have three sets of ∥ forces in equilibrium in the figure, viz.: the internal fluid pressure $= \pi r^2 p$; the external fluid pressure $= \pi r^2 p_a$; while the total stress (tensile) on the small ring, whose area now exposed is $2\pi rt$ (nearly), is $= 2\pi rtp_1$, where t is the thickness of the pipe, and p_1 the tensile stress per unit area induced by the end-pressures (fluid).

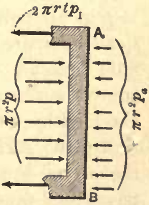

Fig. 469.

For equilibrium, therefore, we may put Σ(hor. comps.) $= 0$; i.e.,

$$\pi r^2 p - \pi r^2 p_a - 2\pi r t p_1 = 0;$$

$$\therefore p_1 = \frac{r(p - p_a)}{2t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(Strictly, the two circular areas sustaining the fluid pressures are different in area, but to consider them equal occasions but a small error.)

Eq. (1) also gives the tension in the central section of a *thin hollow sphere*, under bursting pressure.

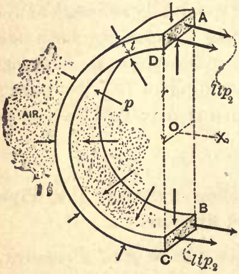

FIG. 470.

CASE II. *Stresses in the longitudinal section of pipe, due to radial fluid pressures.**—Consider free the half (semi-circular) of any length l of the pipe, between two cross-sections. Take an axis X (as in Fig. 470) $\urcorner$ to the longitudinal section which has been made. Let p_2 denote the tensile stress (per unit area) produced in the narrow rectangles exposed at A and B (those in the half-ring edges, having no X components, are not drawn in the figure). On the internal curved surface the fluid pressure is considered of equal intensity $= p$ at all points (practically true even with liquids, if $2r$ is small compared with the head of water producing p). The fluid pressure on any dF or elementary area of the internal curved surface is $= pdF$. Its X component (see § 416) is obtained by multiplying p by the projection of dF on the vertical plane ABC, and since p is the same for all the dF's of the curved surface, the sum of all the X components of the internal fluid pressures must $= p$ multiplied by the area of rectangle $ABCD$, $= 2rlp$; and similarly the X components of the external atmos. pressures $= 2rlp_a$ (nearly). The tensile stresses ($\parallel$ to X) are equal to $2ltp_2$; hence for equilibrium, $\Sigma X = 0$ gives

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* Analytically this problem is identical with that of the smooth cord on a smooth cylinder, § 169, and is seen to give the same result.

$$2ltp_2 - 2rlp + 2rlp_a = 0;$$

$$\therefore p_2 = \frac{r(p - p_a)}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This tensile stress, called *hoop tension*, p_2, opposing rupture by longitudinal tearing, is seen to be double the tensile stress p_1 induced, under the same circumstances, on the annular cross-section in Case I. Hence eq. (2), and not eq. (1), should be used to determine a safe value for the thickness of metal, t, or any other one unknown quantity involved in the equation.

For safety against rupture, we must put $p_2 = T'$, a safe tensile stress per unit area for the material of the pipe or tube (see §§ 195 and 203);

$$\therefore t = \frac{r(p - p_a)}{T'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(For a *thin hollow sphere*, t may be computed from eq. (1); that is, need be only half as great as with the cylinder, other things being equal.)

EXAMPLE.—A pipe of twenty inches internal diameter is to contain water at rest under a head of 340 feet; required the proper thickness, if of cast-iron.

340 feet of water measures 10 atmospheres, so that the internal fluid pressure is 11 atmospheres; but the external pressure p_a being one atmos., we must write (inch, lb., sec.)

$$(p - p_a) = 10 \times 14.7 = 147.0 \text{ lbs. per sq. in., and } r = 10 \text{ in.,}$$

while (§ 203) we may put $T' = \frac{1}{2}$ of $9000 = 4500$ lbs. per sq. in.; whence

$$t = \frac{10 \times 147}{4500} = 0.326 \text{ inches.}$$

But to insure safety in handling pipes and imperviousness to the water, a somewhat greater thickness is adopted in practice than given by the above theory.

Thus, Weisbach recommends (as proved experimentally also) for

$$\text{Not homogen.}\left\{\begin{array}{lll} \text{Pipes of sheet iron,} & t = [0.00172\,rA + 0.12] & \text{inches;} \\ \text{“ “ cast “} & t = [0.00476\,rA + 0.34] & \text{“} \\ \text{“ “ copper} & t = [0.00296\,rA + 0.16] & \text{“} \\ \text{“ “ lead} & t = [0.01014\,rA + 0.21] & \text{“} \\ \text{“ “ zinc} & t = [0.00484\,rA + 0.16] & \text{“} \end{array}\right.$$

in which $t =$ thickness in inches, $r =$ radius in inches, and A $=$ excess of internal over external fluid pressure (i.e., $p - p_a$) expressed in *atmospheres.*

For instance, for the example just given, we should have (cast-iron)

$$t = .00476 \times 10 \times 10 + 0.34 = 0.816 \text{ inches.}$$

If the pipe is subject to “water-ram” (§ 422) the strength should be much greater. To provide against “water-ram,” Mr. J. T. Fanning, on p. 453 of his “Hydraulic and Water-supply Engineering,” advises adding 230 feet to the static head in computing the thickness of cast-iron pipes.

For *thick hollow cylinders* see Rankine's Applied Mechanics, p. 290, and Cotterill's Applied Mechanics, p. 403.

427. Collapsing of Tubes under Fluid Pressure. (Cylindrical boiler-flues, for example.)—If the external exceeds the internal fluid pressure, and the thickness of metal is small compared with the diameter, the slightest deformation of the tube or pipe gives the external pressure greater capability to produce a further change of form, and hence possibly a final collapse; just as with long columns (§ 303) a slight bending gives great advantage to the terminal forces. Hence the theory of § 426 is inapplicable. According to Sir Wm. Fairbairn's experiments (1858) a thin wrought-iron cylindrical (circular) tube will not collapse until the excess of external over internal pressure is

$$p(\text{in lbs. per sq. in.}) = 9672000\frac{t^2}{ld}. \quad . \quad . \ (1) \ . \quad . \ (\text{not homog.})$$

(t, l, and d must all be expressed in the same linear unit.) Here $t =$ thickness of the wall of the tube, d its diameter, and l its length; the ends being understood to be so supported as to preclude a local collapse.

EXAMPLE.—With $l = 10$ ft. $= 120$ inches, $d = 4$ in., and $t = \frac{1}{10}$ inch, we have

$$p = 9672000\left[\frac{1}{100} \div (120 \times 4)\right] = 201.5 \text{ lbs. per sq. inch.}$$

For safety, $\frac{1}{5}$ of this, viz. 40 lbs. per sq. inch, should not be exceeded; e.g., with 14.7 lbs. internal and 54.7 lbs. external.

[NOTE.—For simplicity the power of the thickness used in eq. (1) above has been given as 2.00. In the original formula it is 2.19, and then all dimensions must be expressed in *inches*. A discussion of the experiments of Mr. Fairbairn will be found in a paper read by Prof. Unwin before the Institute of Civ. Engineers (Proceedings, vol. xlvi.). See also Prof. Unwin's "Machine Design," p. 66. It is contended by some that in the actual conditions of service, boiler-flues are subjected to such serious straining actions due to unequal expansion of the connecting parts as to render the above formula quite unreliable, thus requiring a large allowance in its application.]

CHAPTER II.

HYDROSTATICS (*Continued*)—PRESSURE OF LIQUIDS IN TANKS AND RESERVOIRS.

428. Body of Liquid in Motion, but in Relative Equilibrium.—By relative equilibrium it is meant that the particles are not changing their relative positions, i.e., are not moving among each other. On account of this relative equilibrium the following problems are placed in the present chapter, instead of under the head of *Hydrodynamics*, where they strictly belong. As *relative equilibrium* is an essential property of rigid bodies, we may apply the equations of motion of rigid bodies to bodies of liquid in relative equilibrium.

CASE I. *All the particles moving in parallel right lines with equal velocities; at any given instant* (i.e., a motion of *translation.*)—If the common velocity is *constant* we have *a uniform translation*, and all the forces acting on any one particle are balanced, as if it were not moving at all (according to Newton's Laws, § 54); hence the relations of internal pressure, free surface, etc., are the same as if the liquid were at rest. Thus, Fig. 471, if the liquid in the moving tank is at rest relatively to the tank at a given instant, with *its free surface horizontal*, and the motion of the tank be one of translation with a uniform velocity, the liquid will remain in this condition of relative rest, as the motion proceeds.

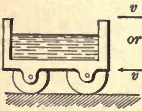

FIG. 471.

But if the velocity of the tank is *accelerated* with a *constant acceleration* $=\bar{p}$ (this symbol must not be confused with p for pressure), the free surface will begin to oscillate, and finally come to relative equilibrium at some angle α with the horizontal, which is thus found, when the motion is horizontal. See Fig. 472, in which the position and value of α are the same, whether the motion is uniformly accelerated from left to right or uniformly retarded from right to left. Let O be the lowest point of the free surface, and Ob a small prism of the liquid with its axis horizontal, and of length $= x$; nb is a vertical prism of length $= z$, and extending from the extremity of Ob to the free surface. The pressure at both O and n is $p_a =$ atmos. pres. Let the area of cross-section of both prisms be $= dF$.

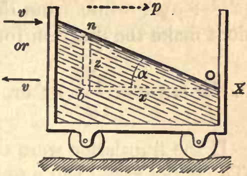

FIG. 472.

Now since Ob is being accelerated in direction X (horizont.), the difference between the forces on its two ends, i.e., its ΣX, must = its mass × accel. (§ 109).

$$\therefore p_b dF - p_a dF = [x dF \cdot \gamma \div g]\bar{p}. \quad . \quad . \quad . \quad (1)$$

($\gamma =$ heaviness of liquid; $p_b =$ press. at b); and since the vertical prism nb has no vertical acceleration, the Σ(vert. compons.) for it must = 0.

$$\therefore p_b dF - p_a dF - z dF \cdot \gamma = 0. \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2),

$$\frac{x\gamma}{g} \cdot \bar{p} = z\gamma; \quad \therefore \frac{z}{x} = \frac{\bar{p}}{g}. \quad . \quad . \quad . \quad . \quad (3)$$

Hence On is a right line, and therefore

$$\tan \alpha, \text{ or } \frac{z}{x}, = \frac{\bar{p}}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

[Another, and perhaps more direct, method of deriving this result is to consider free a small particle of the liquid lying in the surface. The forces acting on this particle are two: the first its weight $= dG$; and the second the resultant action of its immediate neighbor-particles. Now this latter force (pointing obliquely upward) must be normal to the free surface of the liquid, and therefore must make the unknown angle α with the vertical. Since the particle has at this instant a rectilinear accelerated motion in a horizontal direction, the resultant of the two forces mentioned must be horizontal and have a value = mass × acceleration. That is, the diagonal formed on the two forces must be horizontal and have the value mentioned, $= (dG \div g)\bar{p}$; while from the nature of the figure (let the student make the diagram for himself) it must also $= dG \tan \alpha$.

$$\therefore dG \tan \alpha = \frac{dG}{g}.\bar{p}; \text{ or, } \tan \alpha = \frac{\bar{p}}{g}. \quad . \quad . \quad . \quad . \quad \text{Q. E. D.]}$$

If the translation were *vertical*, and the acceleration *upward* [i.e., if the vessel had a uniformly accelerated upward motion or a uniformly retarded downward motion], the free surface would be horizontal, but the pressure at a depth $= h$ below the surface instead of $p = p_a + h\gamma$ would be obtained as follows: Considering free a small vertical prism of height $= h$ with upper base in the free surface, and putting Σ(vert. compons.) $=$ mass $\times$ acceleration, we have

$$dF.p - dF.p_a - hdF.\gamma = \frac{hdF.\gamma}{g}.\bar{p};$$

$$\therefore p = p_a + h\gamma\left[\frac{g + \bar{p}}{g}\right]. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the acceleration is downward (not the velocity necessarily) we make $\bar{p}$ negative in (5). If the vessel falls freely, $\bar{p} = -g$ and $\therefore p = p_a$, in all parts of the liquid.

Query: Suppose $\bar{p}$ downward and $> g$.

CASE II. *Uniform Rotation about a Vertical Axis.*—If the narrow vessel in Fig. 473, open at top and containing a liquid, be kept rotating at a uniform angular velocity ω (see § 110) about a vertical axis Z, the liquid after some oscillations will be brought (by friction) to relative equilibrium (rotating about Z, as if rigid). Required the form of the free surface (evidently a surface of revolution) at each point of which we know $p = p_a$.

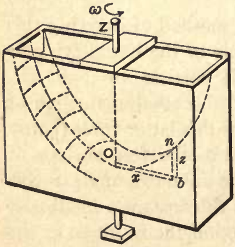

FIG. 473.

Let O be the intersection of the axis Z with the surface, and n any point in the surface; b being a point vertically under n and in same horizontal plane as O. Every point of the small right prism nb (of altitude $= z$ and sectional area dF) is describing a horizontal circle about Z, and has therefore no vertical acceleration. Hence for this prism, free, we have $\Sigma Z = 0$; i.e.,

$$dF.p_b - dF.p_a - zdF.\gamma = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Now the horizontal right prism Ob (call the direction $O \dots b$, X) is rotating uniformly about a vertical axis through one extremity, as if it were a rigid body. Hence the forces acting on it must be equivalent to a single horizontal force, $-\omega^2 M\bar{\rho}$, (§122a,) coinciding in direction with X. [$M =$ mass of prism $=$ its weight $\div g$, and $\bar{\rho} =$ distance of its centre of gravity from O; here $\bar{\rho} = \frac{1}{2}x = \frac{1}{2}$ length of prism]. Hence the ΣX of the forces acting on the prism Ob must $= -\omega^2 \dfrac{xdF}{g}\gamma\frac{1}{2}x$. But the forces acting on the two ends of this prism are their own X components, while the lateral pressures and the weights of its particles have no X compons.;

$$\therefore dF.p_a - dF.p_b = \frac{-\omega^2 x^2 dF.\gamma}{2g}. \quad . \quad . \quad (2)$$

From (1) and (2) we have

$$z = \frac{(\omega x)^2}{2g} = \frac{v^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $v = \omega x =$ linear velocity of the point n in its circular path.

[As in Case I, we may obtain the same result by considering a single surface particle free, and would derive for the resultant force acting upon it the value dG tan α in a horizontal direction and intersecting the axis of rotation. But here α is different for particles at different distances from the axis, tan α being the $\dfrac{dz}{dx}$ of the curve On. As the particle is moving uniformly in a circle the resultant force must point toward the centre of the circle, i.e., horizontally, and have a value $\frac{dG}{g}\cdot\frac{v^2}{x}$, where x is the radius of the circle [§ 74, eq. (5)];

$$\therefore dG \tan\alpha = \frac{dG}{g}\frac{(\omega x)^2}{x};\ \text{or } \tan\alpha, = \frac{dz}{dx}, = \frac{\omega^2 x}{g};$$

$$\therefore \int_0^z dz = \frac{\omega^2}{g}\int_0^x x\,dx;\ \text{or, } z = \frac{\omega^2}{g}\cdot\frac{x^2}{2}.\ \ .\ \ \text{Q. E. D.}$$

Hence any vertical section of the free surface through the axis of rotation Z is a parabola, with its axis vertical and vertex at O; i.e., the free surface is a *paraboloid of revolution*, with Z as its axis. Since ωx is the linear velocity v of the point b in its circular path, $z =$ "height due to velocity" v [§ 52].

EXAMPLE.—If the vessel in Fig. 473 makes 100 revol. per minute, required the ordinate z at a horizontal distance of $x =$ 4 inches from the axis (ft.-lb.-sec. system). The angular velocity $\omega = [2\pi\ 100 \div 60]$ *radians* per sec. [N. B.—A *radian* = the angular space of which 3.1415926 . . . make a half-revol., or angle of 180°]. With $x = \frac{1}{3}$ ft. and $g = 32.2$,

$$z = \frac{\omega^2 x^2}{2g} = \left(\frac{10\pi}{3}\right)^2\left(\frac{1}{3}\right)^2\frac{1}{64.4} = 0.188 \text{ ft.} = 2\tfrac{1}{4} \text{ inches,}$$

and the pressure at b (Fig. 471) is (now use inch, lb., sec.)

$$p_b = p_a + z\gamma = 14.7 + 2\tfrac{1}{4} \times \frac{62.5}{1728} = 14.781 \text{ lbs. per sq. in.}$$

Prof. Mendelejeff of Russia has recently utilized the fact announced as the result of this problem, for forming perfectly true paraboloidal surfaces of plaster of Paris, to receive by galvanic process a deposit of metal, and thus produce specula of exact figure for reflecting telescopes. The vessel containing the liquid plaster is kept rotating about a vertical axis at the proper uniform speed, and the plaster assumes the desired shape before solidifying. A fusible alloy, melted, may also be placed in the vessel, instead of liquid plaster.

Remark.—If the vessel is quite full and closed on top, except at O' where it communicates by a stationary pipe with a reservoir, Fig. 474, the free surface cannot be formed, but the pressure at any point in the water is just the same during uniform rotation, as if a free surface were formed with vertex at O;

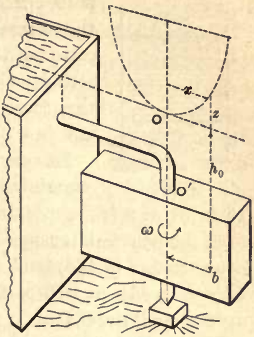

Fig. 474.

i.e., $p_b = p_a + (h_0 + z)\gamma$. . (4)

See figure for h_0 and z. (In subsequent paragraphs of this chapter the liquid will be at rest.)

428a. Pressure on the Bottom of a Vessel containing Liquid at Rest.—If the bottom of the vessel is plane and horizontal, the intensity of pressure upon it is the same at all points, being

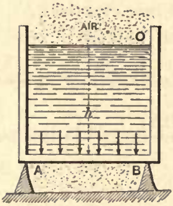

Fig. 475.

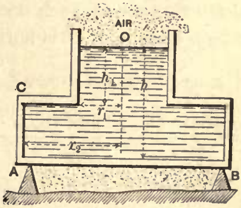

Fig. 476.

$p = p_a + h\gamma$ (Figs. 475 and 476), and the pressures on the elements of the surface form a set of parallel (vertical) forces. This is true even if the side of the vessel overhangs, Fig. 476, the resultant fluid pressure on the bottom in both cases being

$$P = Fp - Fp_a = Fh\gamma. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(Atmospheric pressure is supposed to act under the bottom.) It is further evident that if the bottom is a rigid homogeneous plate and has no support at its edges, it may be supported at a single point (Fig. 477), which in this case (horizontal plate) is its centre of gravity. This point is called the **Centre of Pressure**, or the point of application of the resultant of all the fluid pressures acting on the plate. The present case is such that these pressures reduce to a single resultant, but this is not always practicable.

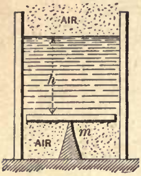

FIG. 477.

EXAMPLE.—In Fig. 476 (cylindrical vessel containing water), given $h = 20$ ft., $h_1 =$ 15 ft., $r_1 = 2$ ft., $r_2 = 4$ ft., required the pressure on the bottom, the vertical tension in the cylindrical wall CA, and the hoop tension (§ 426) at C. (Ft., lb., sec.) Press. on bottom $= Fh\gamma = \pi r_2^2 h\gamma = \pi 16 \times 20 \times 62.5 = 62857$ lbs.; while the upward pull on $CA =$

$$(\pi r_2^2 - \pi r_1^2)h_1\gamma = \pi(16 - 4)15 \times 62.5 = 35357 \text{ lbs.}$$

If the vertical wall is $t = \frac{1}{10}$ inch thick at C this tension will be borne by a ring-shaped cross-section of area $= 2\pi r_2 t$ (nearly) $= 2\pi 48 \times \frac{1}{10} = 30.17$ sq. inches, giving $(35357 \div 30.17) =$ about 1200 lbs. per sq. inch tensile stress (vertical).

The *hoop* tension at C is horizontal and is

$$p'' = r_2(p - p_a) \div t \text{ (see § 426), where } p = p_a + h_1\gamma;$$

$$\therefore p'' = \frac{48 \times 15 \times 12 \times (62.5 \div 1728)}{\frac{1}{10}} = 3125 \text{ lbs. per sq. in.}$$

(using the inch and pound).

429. Centre of Pressure.—In subsequent work in this chapter, since the atmosphere has access both to the free surface of liquid and to the outside of the vessel walls, and p_a would cancel out in finding the resultant fluid pressure on any elementary area dF of those walls, we shall write:

The resultant fluid pressure on any dF of the vessel wall is normal to its surface and is $dP = pdF = z\gamma dF$, in which z is the vertical distance of the element below the free surface of the liquid (i.e., $z =$ the "head of water"). If the surface pressed on is plane, these elementary pressures form a system of parallel forces, and may be replaced by a single resultant (if the plate is rigid) which will equal their sum, and whose point of application, called the **Centre of Pressure**, may be located by the equations of § 22, put into calculus form.

If the surface is *curved* the elementary pressures form a system of forces in space, and hence (§ 38) cannot in general be reduced to a single resultant, but to *two*, the point of application of one of which is arbitrary (viz., the arbitrary origin, § 38).

Of course, the object of replacing a set of fluid pressures by a single resultant is for convenience in examining the equilibrium, or stability, of a rigid body the forces acting on which include these fluid pressures. As to their effect in *distorting* the rigid body, the fluid pressures must be considered in their true positions (see example in § 264), and cannot be replaced by a resultant.

430. Resultant Liquid Pressure on a Plane Surface forming Part of a Vessel Wall. Co-ordinates of the Centre of Pressure.—Fig. 478. Let AB be a portion (of any shape) of a plane surface at any angle with the horizontal, sustaining liquid pressure. Prolong the plane of AB till it intersects the free surface of the liquid. Take this intersection as an axis Y, O being any point on Y. The axis X, ⏋ to Y, lies in the given plane. Let $\alpha =$ angle between the plane and the free surface. Then x and y are the co-ordinates of any elementary area dF of the surface, referred to X and Y. $z =$ the "head of water," below the free surface, of any dF. The pressures are parallel.

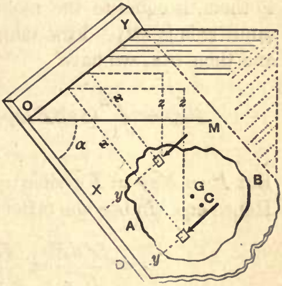

FIG. 478.

The normal pressure on any $dF = z\gamma dF$; hence the *sum of these*, = their resultant,

$$= P, = \gamma\int z dF = F\bar{z}\gamma \; ; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $\bar{z}$ = the "mean z," i.e., the z of the centre of gravity G of the plane figure AB, and F = total area of AB [$F\bar{z} = \int z dF$, from eq. (4), § 23]. γ = heaviness of liquid (see § 409).

That is, *the total liquid pressure on a plane figure is equal to the weight of an imaginary prism of the liquid having a base = area of the given figure and an altitude = vertical depth of the centre of gravity of the figure below the surface of the liquid.* For example, if the figure is a rectangle with one base (length = b) in the surface, and lying in a vertical plane,

$$P = bh \,.\, \tfrac{1}{2}h\gamma = \tfrac{1}{2}bh^2\gamma.$$

Evidently, if the altitude be increased, P varies as its *square.*

From (1) it is evident that the total pressure *does not depend on the horizontal extent of the water in the reservoir.*

Now let x_c and y_c denote the co-ordinates, in plane YOX, of the *centre of pressure*, C, or *point of application of the resultant pressure* P, and apply the principle that the sum of the moments of each of several parallel forces, about an axis $\perp$ to them, is equal to the moment of their resultant about the same axis [§ 22]. First taking OY as an axis of moments, and then OX, we have

$$Px_c = \int_A^B (z\gamma dF)x, \quad \text{and} \quad Py_c = \int_A^B (z\gamma dF)y. \quad . \quad (2)$$

But $P = F\bar{z}\gamma = F\bar{x}(\sin\alpha)\gamma$, and the z of any $dF = x \sin \alpha$. Hence eqs. (2) become (after cancelling the constant, $\gamma \sin \alpha$)

$$x_c = \frac{\int x^2 dF}{F\bar{x}} = \frac{I_Y}{F\bar{x}}, \quad \text{and} \quad y_c = \frac{\int xy dF}{F\bar{x}}; \quad . \quad . \quad (3)$$

in which I_Y = the "*mom. of inertia*" of the plane figure referred to Y (see § 85). [N. B.—The centre of pressure as thus found is identical with the *centre of oscillation* (§ 117) and the *centre of percussion* [§ 113] of a thin homogeneous plate, referred to axes X and Y, Y being the axis of suspension.]

Evidently, if the plane figure is vertical $\alpha = 90°$, $x = z$ for all dF's, and $\bar{x} = \bar{z}$. It is also noteworthy that the position of the centre of pressure is independent of α.

NOTE.—Since the pressures on the equal dF's lying in any horizontal strip of the plane figure form a set of equal parallel forces *equally spaced along the strip*, and are therefore equivalent to their sum applied in the *middle* of the strip, it follows that for rectangles and triangles with horizontal bases, the centre of pressure must lie on the straight line on which the middles of all horizontal strips are situated.

431. Centre of Pressure of Rectangles and Triangles with Bases Horizontal.—Since all the dF's of one horizontal strip have the same x, we may take the area of the strip for dF in the summation $\int x^2 dF$. Hence for the *rectangle AB*, Fig 479, we have from eq. (3), § 430, with $dF = bdx$,

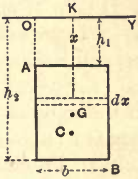

FIG. 479.

$$x_c = \overline{KC} = \frac{b\int_{h_1}^{h_2} x^2 dx}{b(h_2 - h_1)\dfrac{h_1 + h_2}{2}} = \frac{2}{3} \cdot \frac{h_2^3 - h_1^3}{h_2^2 - h_1^2}; \quad (1)$$

while (see note, § 430) $y_c = \frac{1}{2}b$.

When the upper base lies in the surface, $h_1 = 0$, and $x_c = \frac{2}{3}h_2 = \frac{2}{3}$ *of the altitude.*

For a *triangle with its base horizontal and vertex up*, Fig. 480, the length u of a horizontal strip is variable and $dF = udx$. From similar triangles $u = \dfrac{b}{h_2 - h_1}(x - h_1)$; therefore

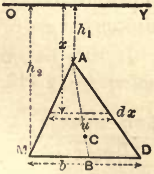

FIG. 480.

$$x_c = \frac{\int_{h_1}^{h_2} x^2 dF}{F\bar{x}} = \frac{\dfrac{b}{h_2 - h_1}\int_{h_1}^{h_2} x^2(x - h_1)dx}{\frac{1}{2}b(h_2 - h_1)[h_1 + \frac{2}{3}(h_2 - h_1)]}.$$

$$\text{But } \int_{h_1}^{h_2} x^2(x - h_1)dx = \left[\left(\frac{x^4}{4} - h_1\frac{x^3}{3}\right)\right]_{h_1}^{h_2}$$

$$= \tfrac{1}{12}(3h_2^4 + h_1^4 - 4h_1h_2^3)$$

$$= \tfrac{1}{12}(h_2 - h_1)^2(3h_2^2 + 2h_1h_2 + h_1^2);$$

$$\therefore x_c = \tfrac{1}{2} \cdot \frac{3h_2^2 + 2h_1h_2 + h_1^2}{2h_2 + h_1}. \quad . \quad . \quad . \quad . \quad (2)$$

H.T. Cory.

Also, since the centre of pressure must lie on the line *AB* joining the vertex to the middle of base (see note, § 430), we easily determine its position.

Evidently for $h_1 = 0$, i.e., when the vertex is in the surface, $x_0 = \frac{2}{3}h_2$. Similarly, *for a triangle with base horizontal and vertex down*, Fig. 481, we find that

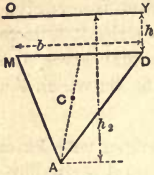

FIG. 481.

$$x = \frac{1}{2}\cdot\frac{3h_1^2 + 2h_1h_2 + h_2^2}{2h_1 + h_2}. \quad . \quad (3)$$

If the base is in the surface, $h_1 = 0$ and (3) reduces to $x_0 = \frac{1}{2}h_2$.

It is to be noticed that in the case of the triangle the value of x_0 is the same whatever be its shape, so long as h_1 and h_2 remain unchanged and the base is horizontal. If the base is not horizontal, we may easily, by one horizontal line, divide the triangle into two triangles whose bases are horizontal and whose combined areas make up the area of the first. The resultant pressure on each of the component triangles is easily found by the foregoing principles, as also its point of application. The resultant of the two parallel forces so determined will act at some point on the line joining the centres of pressures of the component triangles, this point being easily found by the method of moments, while the amount of this final resultant pressure is the sum of its two components, since the latter are parallel. An instance of this procedure will be given in Example 3 of § 433. Similarly, the rectangle of Fig. 479 may be distorted into an oblique parallelogram with horizontal bases without affecting the value of x_0, nor the amount of resultant pressure, so long as h_1 and h_2 remain unchanged.

432. Centre of Pressure of Circle.—Fig. 482. It will lie on the vertical diameter. Let $r =$ radius. From eq. (3), § 413,

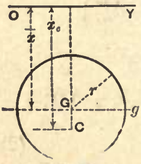

FIG. 482.

$$x_0 = \frac{I_Y}{F\bar{x}}; \; = \frac{I_g + F\bar{x}^2}{F\bar{x}} = \frac{\frac{1}{4}\pi r^4 + \pi r^2\bar{x}^2}{\pi r^2\bar{x}}.$$

(See eq. (4), § 88, and also § 91.)

$$\therefore x_0 = \bar{x} + \frac{1}{4}\cdot\frac{r^2}{\bar{x}}. \quad . \quad . \quad . \quad (4)$$

433. Examples.—It will be noticed that although the total pressure on the plane figure depends for its value upon the head, $\bar{z}$, of the centre of gravity, its point of application is always *lower* than the centre of gravity.

EXAMPLE 1.—If 6 ft. of a vertical sluice-gate, 4 ft. wide, Fig. 483, is below the water-surface, the total water pressure against it is (ft., lb., sec.; eq. (1), § 430)

$$P = F\bar{z}\gamma = 6 \times 4 \times 3 \times 62.5 = 4500 \text{ lbs.},$$

and (so far as the pressures on the vertical posts on which the gate slides are concerned) is equivalent to a single horizontal force of that value applied at a distance $x_c = \frac{2}{3}$ of $6 = 4$ ft. below the surface (§ 431).

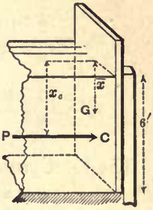

FIG. 483.

EXAMPLE 2.—To (begin to) lift the gate in Fig. 483, the gate itself weighing 200 lbs., and the coefficient of friction between the gate and posts being $f = 0.40$ (abstract numb.) (see § 156), we must employ an upward vertical force at least

$$= P' = 200 + 0.40 \times 4500 = 2000 \text{ lbs.}$$

EXAMPLE 3.—It is required to find the resultant hydrostatic pressure on the trapezoid in Fig. 483*a* with the dimensions there given and its bases horizontal; also its point of application, i.e., the centre of pressure of the plane figure in the position there shown. From symmetry the C. of P. will be in the middle vertical of the figure, as also that of the rectangle $BCFE$, and that of the two triangles ABE and CDF taken together (conceived to be shifted horizontally so that CF and BE coincide on the middle vertical, thus forming a single triangle of 5 ft. base, and having the same total pressure and C. of P. as the two actual triangles taken together). Let P_1 = the total pressure, and x_c' refer to the C. of P., for the rectangle; P_2 and x_c'', for the 5 ft. tri-

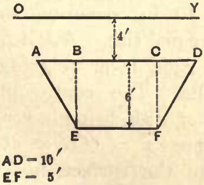

FIG. 483*a*.

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angle; $h_1 = 4$ ft. and $h_2 = 10$ ft. being the same for both. Then from eq. (1), § 430, we have (with the ft., lb., and sec.)

$$P_1 = 30 \times 7\gamma = 210\gamma; \quad \text{and} \quad P_2 = \tfrac{1}{2} \times 6 \times 5 \times 6\gamma = 90\gamma;$$

while from eqs. (1) and (3) of § 431 we have also (respectively)

$$x_c' = \frac{2}{3} \cdot \frac{1000 - 64}{100 - 16} = \frac{2}{3} \cdot \frac{936}{84} = 7.438 \text{ feet};$$

$$x_c'' = \frac{1}{2} \cdot \frac{48 + 80 + 100}{8 + 10} = \frac{228}{2 \times 18} = 6.333 \text{ feet}.$$

The total pressure on the trapezoid, being the resultant of P_1 and P_2, has an amount $= P_1 + P_2$ (since they are parallel), and has a lever-arm x_c about the axis OY to be found by the principle of moments, as follows:

$$x_c = \frac{P_1 x_c' + P_2 x_c''}{P_1 + P_2} = \frac{(210 \times 7.438 + 90 \times 6.33)\gamma}{(210 + 90)\gamma} = 7.09 \text{ ft.}$$

The total hydrostatic pressure on the trapezoid is (for fresh water)

$$P = P_1 + P_2 = [210 + 90]\, 62.5 = 18750 \text{ lbs.}$$

Example 4.—Required the horizontal force P', Fig. 484, to be applied at N (with a leverage of $a' = 30$ inches about the fulcrum M) necessary to (begin to) lift the circular disk AB of radius $r = 10$ in., covering an opening of equal size. $NMAB$ is a single rigid lever weighing $G' = 210$ lbs. The centre of gravity, G, of disk, being a vertical distance $\bar{z} = O'G = 40$ inches from the surface, is 50 inches (viz., the sum of $OM = k = 20''$ and $MG = 30''$) from axis OY; i.e., $\bar{x} = 50$ inches. The centre of gravity of the whole lever is a horizontal distance b', $= 12$ inches, from M.

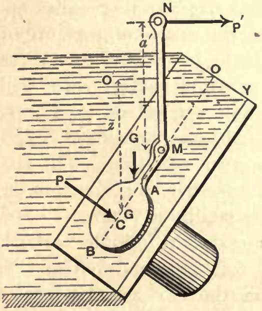

FIG. 484.

For impending lifting we must have, for equilibrium of the lever,

$$P'a' = G'b' + P(x_c - k); \quad . \quad . \quad . \quad . \quad (1)$$

where P = total water pressure on circular disk, and $x_c = OC$. From eq. (1), § 430, (using inch, lb., and sec.,)

$$P = F\bar{z}\gamma = \pi r^2\bar{z}\gamma = \pi 100 \times 40 \times \frac{62.5}{1728} = 454.6 \text{ lbs.}$$

$$\text{From §432, } x_c = \overline{OC} = \bar{x} + \frac{1}{4}\frac{r^2}{\bar{x}} = 50 + \frac{1}{4}\cdot\frac{100}{50} = 50.5 \text{ in.}$$

$$\therefore P' = \frac{1}{a'}\Big[G'b' + P(x_c - k)\Big]$$

$$= \frac{1}{30}[210 \times 12 + 454.6 \times 30.5] = 546 \text{ lbs.}$$

434. Example of Flood-gate.—Fig. 485. Supposing the rigid double gate AD, 8 ft. in total width, to have four hinges; two at e, and two at f, 1 ft. from top and bottom of water channel; required the pressures upon them, taking dimensions from the figure (ft., lb., sec.).

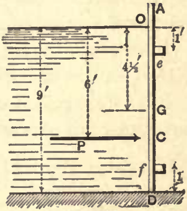

FIG. 485.

$$\text{Wat. press.} = P = F\bar{z}\gamma$$

$$= 72 \times 4\tfrac{1}{2} \times 62.5 = 20250$$

pounds, and its point of application (cent. of press.) is a distance $x_c = \frac{2}{3}$ of $9' = 6'$ from O (§ 431). Considering the whole gate free and taking moments about e, we shall have

$$(\text{press. at } f) \times 7' = 20250 \times 5; \quad \therefore \text{ press. at } f = 14464 \text{ lbs.}$$

(half on each hinge at f), and

$$\therefore \text{ press. at } e = P - \text{press. at } f = 5875 \text{ lbs.}$$

(half coming on each hinge).

If the two gates do not form a single rigid body, and hence are not in the same plane when closed, a wedge-like or toggle-joint action is induced, producing much greater thrusts against the hinges, and each of these thrusts is not ⌉ to the plane of the corresponding gate. Such a case forms a good exercise for the student.

435. Stability of a Vertical Rectangular Wall against Water Pressure on One Side.—Fig. 486. All dimensions are shown in the figure, except l, which is the length of wall ⌉ to paper. Supposing the wall to be a single rigid block, its weight $G' = b'h'l\gamma'$ (γ' being its heaviness (§ 7), and l its length). Given the water depth $= h$, required the proper width b' for stability. For proper security: *First*, the resultant of G' and the water-pressure P must fall within the base BD (or, which amounts to the same thing), the moment of G' about D, the outer *toe* of the wall, must be numerically greater than that of P; and

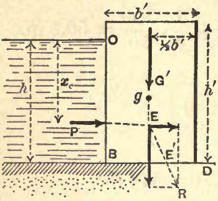

FIG. 486.

Secondly, P must be less than the sliding friction fG' (see § 156) on the base BD.

Thirdly, the maximum pressure per unit of area on the base must not exceed a safe value (compare § 348).

Now $P = F\bar{z}\gamma = hl\dfrac{h}{2}\gamma = \dfrac{1}{2}h^2l\gamma$ ($\gamma =$ heaviness of water); and $x_c = \frac{2}{3}h$.

Hence for stability against tipping about D,

$$P\tfrac{1}{3}h \text{ must be } < G'\tfrac{1}{2}b'; \quad \text{i.e., } \tfrac{1}{6}h^3l\gamma < \tfrac{1}{2}b'^2h'l\gamma'; \quad . \quad (1)$$

while, as to sliding on the base,

$$P \text{ must be } < fG'; \quad \text{i.e., } \tfrac{1}{2}h^2l\gamma < fb'h'l\gamma'. \quad . \quad . \quad (2)$$

As for values of the coefficient of friction, f, on the base of wall, Mr. Fanning quotes the following among others, from various authorities:

For point-dressed granite on dry clay,	$f = 0.51$
" " " " " moist clay,	0.33
" " " " " gravel,	0.58
" " " " " smooth concrete,	0.62
" " " " " similar granite,	0.70
For dressed hard limestone on like limestone,	0.38
" " " " brickwork,	0.60
For common bricks on common bricks,	0.64

To satisfactorily investigate the third condition requires the detail of the next paragraph.

436. Parallelopipedical Reservoir Walls. More Detailed and Exact Solution.—If (1) in the last paragraph were an exact equality, instead of an inequality, the resultant R of P and G' would pass through the corner D, tipping would be impending, and the pressure per unit area at D would be theoretically *infinite*. To avoid this we wish the wall to be wide enough that the resultant R, Fig. 487, may cut BD in such a point, E', as to cause the pressure per unit area, p_m, at D to have a definite safe value (for the pressure p_m at D, or quite near D, will evidently be greater than elsewhere on BD; i.e., it is the maximum pressure to be found on BD). This may be done by the principles of §§ 346 and 362.

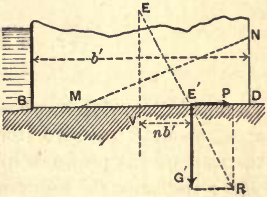

FIG. 487.

First, assume that R cuts BD *outside of the middle third;* i.e., that

$$VE', = nb', > \tfrac{1}{6}b' (\text{or } n > \tfrac{1}{6});$$

where n denotes the ratio of the distance of E' from the middle of the base to the whole width, b', of base. Then the pressure (per unit area) on small equal elements of the base BD (see § 346) may be considered to vary as the ordinates of a triangle MND (the vertex M being within the distance BD), and $\overline{E'D}$ will $= \frac{1}{3}\overline{MD}$; i.e.,

$$\overline{MD} = 3(\tfrac{1}{2} - n)b'.$$

The mean pressure per unit area, on MD,

$$= G' \div (l \,.\, \overline{MD}),$$

and hence the maximum pressure (viz., at D), being double the mean, is

$$p_m = 2G' \div [3b'l(\tfrac{1}{2} - n)]; \quad . \quad . \quad . \quad . \quad (0)$$

and if p_m is to equal C' (see §§ 201 and 203), a safe value for the crushing resistance, per unit area, of the material, we shall have

$$b'l(\tfrac{1}{2} - n)C' = \tfrac{2}{3}G' = \tfrac{2}{3}b'h'l\gamma',$$

$$\therefore \quad n = \frac{1}{2} - \frac{2}{3}\frac{h'\gamma'}{C'}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To find b', knowing n, we put the Σ(moms.) of the G' and P at E, about E', $=$ zero (for the only other forces acting on the wall are the pressures of the foundation against it, along MD; and since the resultant of these latter passes through E', the sum of their moments about E' is already zero); i.e.,

$$G'nb' - P\tfrac{1}{3}h = 0; \quad \text{or,} \quad nb'^2h'l\gamma' = \tfrac{1}{3}h \,.\, \tfrac{1}{2}h^2l\gamma;$$

$$\therefore \quad b' = h \,.\, \sqrt{\frac{h\gamma}{6nh'\gamma'}}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Having obtained b', we must also ascertain if P is $< fG'$, the friction; i.e., if P is $< fb'h'l\gamma'$. If not, b' must be still further increased. (Or, graphically, the resultant of G' and P must not make an angle $> \phi$, the angle of friction, with the vertical.

If n, computed from (1), should prove to be $< \frac{1}{6}$, our first assumption is wrong, and we therefore assume $n < \frac{1}{6}$, and proceed thus:

Secondly, n being $< \frac{1}{6}$ (see §§ 346 and 362), we have a trapezoid of pressures, instead of a triangle, on BD. Let the pressure per unit area at D be p_m (the maximum on base). The whole base now receives pressure, the mean pressure (per unit area) being $= G' \div [b'l]$; and therefore, from § 362, Case I, we have

$$p_m = [6n + 1]\frac{G'}{b'l}; \quad . \quad . \quad . \quad . \quad . \quad (0a)$$

and since, here, $G' = b'h'l\gamma'$, we may write

$$p_m = (6n + 1)h'\gamma'.$$

For safety as to crushing resistance we put

$$6(n + 1)h'\gamma' = C'; \text{ whence } n = \frac{1}{6}\left[\frac{C'}{h'\gamma'} - 1\right]. \quad . \quad . \quad (1a)$$

Having found n from eq. (1a), we determine the proper width of base b' from eq. (2), in case the assumption $n < \frac{1}{6}$ is verified.

Example.—In Fig. 486, let $h' = 12$ ft., $h = 10$ ft., while the masonry weighs ($\gamma' =$) 150 lbs. per cub. ft. Supposing it desirable to bring no greater compressive stress than 100 lbs. per sq. inch ($=$ 14400 lbs. per sq. ft.) on the cement of the joints, we put $C' = 14400$, using the ft.-lb.-sec. system of units.

Assuming $n > \frac{1}{6}$, we use eq. (1), and obtain

$$n = \frac{1}{2} - \frac{2}{3} \cdot \frac{12 \times 150}{14400} = \frac{5}{12},$$

which is $> \frac{1}{6}$; hence the assumption is confirmed, also the propriety of using eq. (1) rather than (1a).

Passing to eq. (2), we have

$$b' = 10 \times \sqrt{\frac{62.5 \times 10}{\frac{5}{2} \times 12 \times 150}} = 3.7 \text{ feet.}$$

But, as regards frictional stability, we find that, with $f = 0.30$, a low value, and $b' = 3.7$ ft. (ft., lb., sec.),

$$\frac{P}{fG'} = \frac{\frac{1}{2}h'^2 l\gamma}{fb'h'l\gamma'} = \frac{100 \times 62.5}{2 \times 0.3 \times 3.7 \times 12 \times 150} = 1.5;$$

which is greater than unity, showing the friction to be insufficient to prevent sliding (with $f = 0.30$); a greater width must therefore be chosen, for frictional stability.

If we make $n = \frac{1}{6}$, i.e., make R cut the base at the outer edge of middle third (§ 362), we have, from eq. (2),

$$b' = 10 \times \sqrt{\frac{62.5 \times 10}{\frac{8}{6} \times 12 \times 150}} = 5.89 \text{ feet};$$

and the pressure at D is now of course well within the safe limit; while as regards friction we find

$$P \div fG' = 0.92, < \text{unity},$$

and therefore the wall is safe in this respect also.

With a width of base = 3.7 feet first obtained, the portion *MD*, Fig. 487, of the base which receives pressure [according to Navier's theory (§ 346)] would be only 0.92 feet in length, or about one sixth of the base, the portion *BM* tending to open, and perhaps actually suffering tension, if capable (i.e., if cemented to a rock foundation), in which case these tensions should properly be taken into account, as with beams (§ 295), thus modifying the results.

It has been considered safe by some designers of high masonry dams, to neglect these possible tensile resistances, as has just been done in deriving $b' = 3.7$ feet; but others, in view of the more or less uncertain and speculative character of Navier's theory, when applied to the very wide bases of such structures, prefer, in using the theory (as the best available), to keep the resultant pressure within the middle third at the base (and also at all horizontal beds above the base), and thus avoid the chances of tensile stresses.

This latter plan is supported by Messrs. Church and Fteley, as engineers of the proposed Quaker Bridge Dam in connection with the New Croton Aqueduct of New York City, in their report of 1887. See § 438.

437. Wall of Trapezoidal Profile. Water-face Vertical.—Economy of material is favored by using a trapezoidal profile, Fig. 488. With this form the stability may be investigated in a corresponding manner. The portion of wall above each horizontal bed should be examined similarly. The weight G' acts through the centre of gravity of the whole mass.

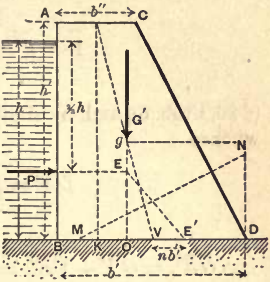

FIG. 488.

Detail.—Let Fig. 488 show the vertical cross-section of a trapezoidal wall, with notation for dimensions as indicated; the portion considered having a length $= l$, ⏋ to the paper. Let $\gamma =$ heaviness of water, γ' that of the masonry (assumed homogeneous), with n as in § 436.

For a *triangle of pressure*, MD, on the base, i.e., with $n > \frac{1}{6}$, or resultant falling outside the middle third (neglecting possibility of tensile stresses on left of M), if the intensity of pressure p_m at D is to $= C'$ (§ 203), we put, as in § 436,

$$b'l[\tfrac{1}{2} - n]C' = \tfrac{2}{3}G', \text{ i.e., } = \tfrac{2}{3}lh' \cdot \tfrac{1}{2}(b' + b'')\gamma',$$

whence

$$n = \frac{1}{2} - \frac{1}{3}\frac{h'\gamma'}{C'} \cdot \frac{b' + b''}{b'}. \quad . \quad . \quad . \quad . \quad (1)'$$

For a *trapezoid of pressure*, i.e. with $n < \frac{1}{6}$, or the resultant of P and G' falling within the middle third, we have, as before (§ 362, Case I),

$$p_m, \text{or } C', = (6n + 1)\frac{G'}{b'l};$$

whence

$$n = \frac{1}{6}\left[\frac{C'b'l}{G'} - 1\right]; \text{ i.e., } n = \frac{1}{6}\left[\frac{2C'b'}{h'\gamma'(b' + b'')} - 1\right]. \quad (1a)'$$

From the geometry of the figure, having joined the middles of the two bases, we have

$$\bar{y}, = \overline{gO}, = \frac{h'}{3} \cdot \frac{b' + 2b''}{b' + b''}$$

(§ 26, Prob. 6), and, by similar triangles, $\overline{OV} : \overline{KV} :: \overline{gO} : h'$, whence

$$\overline{OV} = \frac{\bar{y}}{h'} \cdot \frac{1}{2} [b' - b''] ;$$

$$\therefore \overline{OE'}, = \overline{OV} + \overline{VE'}, = \frac{1}{6} \cdot \frac{(b' + 2b'')(b' - b'')}{b' + b''} + nb' . . (a)$$

The lines of action of G' and P meet at E, and their resultant cuts the base in some point E'. The sum of their moments about E' should be zero, i.e., $P . \frac{1}{3}h = G' . \overline{OE'}$; that is, (see eq. (*a*) above, and eq. (1), § 430,)

$$\tfrac{1}{6}h^3 l\gamma = lh'\gamma'\tfrac{1}{2}(b' + b'')\left[\frac{1}{6} \cdot \frac{(b' + 2b'')(b' - b'')}{b' + b''} + nb'\right] ; \quad (b)$$

i.e., cancelling,

$$h^3\gamma = \tfrac{1}{2} h'\gamma'\big[(b' + 2b'')(b' - b'') + 6nb'(b' + b'')\big]. \qquad (2)'$$

Hence we have two equations for finding two unknowns viz.: (1)′ and (2)′ when $n > \frac{1}{6}$; and (1a)′ and (2)′ when $n < \frac{1}{6}$.

For dams of small height (less than 40 ft., say), if we immediately put $n = \frac{1}{6}$, thus restricting the resultant pressure to the edge of middle third, and solve (2)′ for b', b'' being assumed of some proper value for a coping, foot-walk, or roadway, while h' may be taken enough greater than h to provide against the greatest height of waves, from 2.5 to 6 ft., the value of p_m at D will probably be $< C'$. In any case, for a value of $n =$, or $<$, $\frac{1}{6}$ we put p_m for C' in equation (1a)′ and solve for p_m, to determine if it is no greater than C'.

Mr. Fanning recommends the following values for C' (*in lbs.*

per sq. foot) with coursed rubble masonry laid in strong mortar:

	For Limestone.	Sandstone.	Granite.	Brick.
$C' =$	50,000	50,000	60,000	35,000
Av. heaviness of the masonry in lbs. per cub. ft.	152	132	154	120

As to *frictional resistance*, P must be $< fG'$; i.e.,

$$\tfrac{1}{2}h^2l\gamma < fh'\gamma'\tfrac{1}{2}(b' + b''). \quad . \quad . \quad . \quad . \quad . \quad (3)'$$

If the base is cemented to a *rock foundation* with good material and workmanship throughout, Messrs. Church and Fteley (see § 436) consider that the wall may be treated as amply safe against sliding on the base (or any horizontal bed), provided the other two conditions of safety are already satisfied.

438. Triangular Wall with Vertical Water-face.—Making $b'' = 0$ in the preceding article, the trapezoid becomes a *right triangle*, and the equations reduce to the following:

$$p_m = \frac{2h'\gamma'}{3 - 6n} \text{ for } n > \tfrac{1}{6}, \quad . \quad . \quad . \quad . \quad (1)''$$

and

$$p_m = \tfrac{1}{2}h'\gamma'[6n + 1] \text{ for } n < \tfrac{1}{6} \quad . \quad . \quad . \quad (1a)''$$

(p_m not to exceed C' in any case); while to determine the breadth of base, b', after n is computed [or assumed, for small height of wall], we have from eq. (2)',

$$h^3\gamma = \tfrac{1}{2}h'b'^2\gamma'[6n + 1]. \quad . \quad . \quad . \quad . \quad (2)''$$

Also, for frictional stability,

$$\tfrac{1}{2}h^3l\gamma \text{ must be } < \tfrac{1}{2}fh'b'l\gamma'. \quad . \quad . \quad . \quad . \quad (3)''$$

439. High Masonry Dams.—Although the principle of the arch may be utilized for vertical stone dikes of small height (30 to 50 feet) and small span, for greater heights and spans the formula for hoop tension, § 426 (or rather, here, "hoop compression"), on the vertical radial joints of the horizontal arch rings, Fig. 489, calls for so great a radial thickness of joint in the lower courses, that straight dikes (or "gravity dams") are usually built instead, even where firm rock abutments are available laterally.

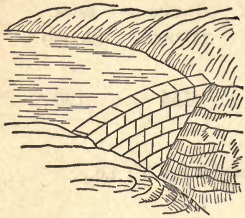

FIG. 489.

For example, at a depth of 100 feet, where the hydrostatic pressure is $h\gamma = 100 \times 62.5 = 6250$ lbs. per sq. ft., if we assume for the voussoirs a (radial, horizontal) thickness $= 4$ ft., with a (horizontal) radius of curvature $r = 100$ feet, we shall find a compression between their vertical radial faces of (ft., lb., sec.)

$$p'' = \frac{r(p - p_a)}{t} = \frac{100 \times 6250}{4} = 156250 \text{ lbs. per sq. ft.},$$

or 1085 lbs. per sq. inch; far too great for safety, even if there were no danger of collapse, the dike being short. If now the thickness is increased, in order to distribute the pressure over a greater surface, we are met by the fact that the formula for "hoop compression" is no longer strictly applicable, the law of distribution of pressure becoming very uncertain; and even supposing a uniform distribution over the joint, the thickness demanded for proper safety against crushing is greater than for a straight dam ("*gravity dam*") at a very moderate depth below the water surface, unless the radius of curvature of arch can be made small. But the smaller the radius the more does the dam encroach on the storage capacity of the reservoir, while in no case, of course, can it be made smaller than half the span.

Another point is, that as masonry is not destitute of elasticity, the longer the span the more unlikely is it that the parts of the arch will "close up" properly, and develop the abutment reactions when the water is first admitted to the reservoir; which should occur if it is to act as an arch instead of by gravity resistance.

For these reasons the engineers of the proposed Quaker Bridge Dam reported unfavorably to the plan of a curved design for that structure, and recommended that a straight dam be built. See reference in § 436. According to their designs this dam is to be 258 feet in height (which exceeds by about 90 feet the height of any dam previously built), about 1400 feet in length at the top, and 216 feet in width at the lowest point of base, joining the bed-rock.

More recently, however (1888), a board of experts, specially appointed for the purpose, having examined a number of different plans, have reported favorably to the adoption of a curved form for the dam, as offering greater resistance under extraordinary circumstances (impact of ice-floes, earthquakes, etc.), on account of its arched form (though resisting by gravity action under usual conditions) than a straight structure; and also as more pleasing in appearance.

Fig. 490 shows the profile of a straight high masonry dam as designed at the present day. Assuming a width $b'' =$ from 6 to 22 feet at the top, and a sufficient h'' (see figure) to exceed the maximum height of waves, the up-stream outline ACM is made nearly vertical and perhaps somewhat concave, while the down-stream profile BDN, by computation or graphical trial, or both, is so formed that *when the reservoir is full* the resultant R, of the weight G of the portion $ABCD$ of masonry above *each* horizontal bed, as CD, and the hydrostatic pressure P on the corresponding up-stream face AC, shall cut the bed CD in such a point E' as not to cause too great compression p_m at the outer edge D (not over 85 lbs. per sq. inch according to M. Krantz in "Reservoir Walls"), p_m being computed by one of the equations [(0) and (0a) of § 436]

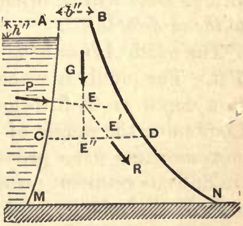

FIG. 490.

$$\left.\begin{array}{l}\text{For } E' \text{ outside the middle third}\\ \quad\text{and neglecting tension}\end{array}\right\} \,.\,. p_m = \frac{2G}{3.\overline{CD}\,.\,l(\frac{1}{2} - n)}; \,(1)'''$$

$$\text{For } E' \text{ inside middle third} \Big\} \,.\,. p_m = \frac{(6n + 1)G}{\overline{CD}\,.\,l}; \quad (1a)'''$$

where $l =$ length of wall $\rceil$ to paper, usually taken = one foot, or one inch, according to the unit of length adopted; for n, see § 436.

Nor, *when the reservoir is empty* and the water pressure lacking, must the weight G resting on each bed, as CD, cut the bed in a point E'' so near the edge C as to produce excessive pressure there (computed as above). The figure shows the general form of profile resulting from these conditions. The masonry should be of such a character, by irregular bonding in every direction, as to make the wall if possible a monolith. For more detail see next paragraph.

440. Quaker Bridge Dam (on the New Croton Aqueduct).—Attempts, by strict analysis, to determine the equation of the curve BN, AM being assumed straight, so as to bring the point E' at the outer edge of the middle third of its joint, or to make the pressure at D constant below a definite joint, have failed, up to the present time; but approximate and tentative methods are in use which serve all practical purposes. As an illustration the method set forth in the report on the Quaker Bridge Dam will be briefly outlined; *this method confines E' to the middle third.*

The width $AB = b''$ is taken $= 22'$ for a roadway, and $h'' =$ 7 ft. The profile is made a vertical rectangle from A down to a depth of 33 ft. below the water surface (*reservoir full*). Combining the weight of this rectangle of masonry with the corresponding water pressure (for a length of wall = one foot), we find the resultant pressure comes a little within the outer edge of the middle third of the base of the rectangle, while p_m is of course small.

The rectangular form of profile might be continued below this horizontal joint, as far as complying with the middle third requirement, and the limitation of pressure-intensity, is concerned; but, not to make the widening of the joints too abrupt in a lower position where it would be absolutely required, a beginning is made at the joint just mentioned by forming a trapezoid between it and a joint 11 ft. farther down, making the lower base of the latter of some trial width, which can be altered when the results to which it gives rise become evident. Having computed the weight of this trapezoid and constructed its line of action through the centre of gravity of the trapezoid, the value of the resultant G of this weight and that of the rectangle is found (by principle of moments or by an equilibrium polygon) in amount and position, and combined with the water pressure of the corresponding 44 ft. of water to form the force R, whose point of intersection with the new joint or bed (lower base of trapezoid) is noted and the value of p_m computed. These should both be somewhat nearer their limits than in the preceding joint. If not, a different width should be chosen, and changed again, if necessary, until satisfactory. Similarly, another layer, 11 ft. in height and of trapezoidal form, is added below and treated in the same way; and so on until in the joint at a depth of 66 ft. from the water surface a width is found where the point E' is very close upon its limiting position, while p_m is quite a little under the limit set for the upper joints of the dam, 8 tons per square foot. For the next three 11 ft. trapezoidal layers the chief governing element is the middle-third requirement, E' being kept quite close to the limit, while the increase of p_m to 7.95 tons per sq. ft. is unobjectionable; also, we begin to move the left-hand edge to the left of the vertical, so that when the reservoir is empty the point E'' shall not be too near the up-stream edge C.

Down to a depth of about 200 ft. the value of p_m is allowed to increase to 10.48 tons per sq. ft., while the position of E' gradually retreats from the edge of its limit. Beyond 200 ft. depth, to prevent a rapid increase of width and consequent extreme flattening of the down-stream curve, p_m is allowed to mount rapidly to 16.63 tons per sq. ft. (= 231 lbs. per sq. in.), which value it reaches at the point N of the base of the dam, which has a width = 216 ft., and is 258 feet below the water surface when the reservoir is full.

The heaviness of the masonry is taken as $\gamma' = 156.25$ lbs. per cubic foot, just $\frac{5}{2}$ of $\gamma = 62.5$ lbs. per cub. foot, the heaviness taken for water.

When the *reservoir is empty*, we have the weight G of the superincumbent mass resting on any bed CD, and applied through the point E''; the pressure per unit area at C can then be computed by eq. $(1a)'''$, § 439, n being the quotient of $(\frac{1}{2}\overline{CD} - CE'') \div \overline{CD}$ for this purpose. In the present case we find E'' to be within middle third at all joints, and the pressures at C to be under the limit.

For further details the reader is referred to the report itself (reprinted in *Engineering News*, January, 1888, p. 20). The graphic results were checked by computation, Wegmann's method, applied to each trapezoid in turn.

441. Earthwork Dam, of Trapezoidal Section.—Fig. 491. It is required to find the conditions of stability of the straight earthwork dam $ABDE$, whose length $= l$, ⊥ to paper, as regards sliding horizontally on the plane AE; i.e., its frictional stability. With the dimensions of the figure, γ and γ' being the heavinesses of the water and earth respectively (see § 7), we have

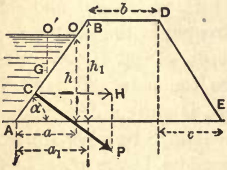

FIG. 491.

$$\text{Weight of dam} = G_1 = \text{vol.} \times \gamma' = lh_1[b + \tfrac{1}{2}(a_1 + c)]\gamma'. \quad (1)$$

$$\text{Resultant water press.} = P = F\bar{z}\gamma = \overline{OA} \times l \times \tfrac{1}{2}h\gamma. \quad . \quad (2)$$

$$\text{Horiz. comp. of } P = H = P \sin\alpha$$

$$= [\overline{OA} \sin\alpha]\tfrac{1}{2}hl\gamma = \tfrac{1}{2}h^2l\gamma. \quad . \quad . \quad . \quad (3)$$

From (3) it is evident that the horizontal component of P is just the same, viz., $= hl \,.\, \frac{1}{2}h\gamma$, as the water pressure would be on a vertical rectangle equal to the vertical projection of OA

and with its centre of gravity at the same depth ($\frac{1}{2}h$). Compare § 416. Also,

$$\text{Vert. comp. of } P = V = P \cos \alpha$$

$$= [\overline{OA} \cos \alpha]\tfrac{1}{2}hl\gamma = \tfrac{1}{2}ahl\gamma, \quad . \quad . \quad . \quad (4)$$

and is the same as the water pressure on the horizontal projection of OA if placed at a depth $= O'G = \frac{1}{2}h$.

For stability against sliding, the horizontal component of P must be less than the friction due to the total vertical pressure on the plane AE, viz., $G_1 + V$; hence if f is the coefficient of friction on AE, we must have $H < f\,[G_1 + V]$, i.e. (see above),

$$\tfrac{1}{2}h^2l\gamma \text{ must be } < f\Big[\, lh_1[b + \tfrac{1}{2}(a_1 + c)]\gamma' + \tfrac{1}{2}ahl\gamma \,\Big]. \quad . \quad (5)$$

However, if the water leak under the dam on the surface AE, so as to exert an upward hydrostatic pressure

$$V' = [a_1 + b + c]lh\gamma,$$

(to make an extreme supposition,) the friction will be only

$$= f[G_1 + V - V'],$$

and (5) will be replaced by

$$H < f[G_1 + V - V']. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Experiment shows (Weisbach) that with $f = 0.33$ computations made from (6) (treated as a bare equality) give satisfactory results.

Example.—(Ft., lb., sec.) With $f = 0.33$, $h = 20$ ft., $h_1 = 22$ ft., $a = 24$ ft., $a_1 = 26.4$ ft., and $c = 30$ ft., we have, making (6) an equality, with $\gamma' = 2\gamma$,

$$\tfrac{1}{2}h^2l\gamma = f\Big[\,\gamma' lh_1\Big(b + \frac{a_1 + c}{2}\Big) + \tfrac{1}{2}ahl\gamma - (a_1 + b + c)lh\gamma\,\Big];$$

$$\therefore\ \tfrac{1}{2}(400) = \tfrac{1}{3}[22(b + 28.2)2 + \tfrac{1}{2}(24 \times 20) - (26.4 + b + 30)20];$$

whence, solving for b, the width of top, $b = 10.3$ feet.

442. Liquid Pressure on Both Sides of a Gate or Rigid Plate.—The sluice-gate AB, for example, Fig. 492, receives a pressure, P_1, from the "head-water" M, and an opposing pressure P_2 from the "tail-water" N. Since these two horizontal forces are not in the same line, though parallel, their resultant R, which $= P_1 - P_2$, acts horizontally in the same plane, but at a distance below $O_1 = u$, which we may find by placing the moment of R about O_1, equal to the algebraic sum of those of P_1 and P_2 about O_1.

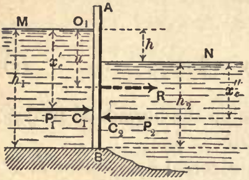

FIG. 492.

$$\therefore \quad Ru = P_1 x_c' - P_2(x_c'' + h). \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \quad u = \frac{[P_1 x_c' - P_2(x_c'' + h)]}{P_1 - P_2}. \quad . \quad . \quad . \quad (2)$$

C_1 and C_2 are the respective centres of pressure of the surfaces O_1B and O_2B, and $u =$ distance of R from O_1, while $h =$ difference of level between head and tail waters. If the surfaces O_1B and O_2B are both rectangular,

$$x_c' = \tfrac{2}{3}h_1 \quad \text{and} \quad x_c'' = \tfrac{2}{3}h_2.$$

EXAMPLE.—Let the dimensions be as in Fig. 493, both surfaces under pressure being rectangular and 8 ft. wide. Then (ft., lb., sec.) $R = P_1 - P_2$, or (§ 430)

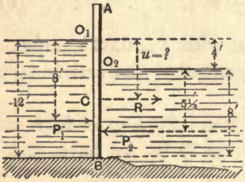

FIG. 493.

$$R = [12 \times 8 \times 6 - 8 \times 8 \times 4]62.5$$

$$= 20000 \text{ lbs.} = 10 \text{ tons};$$

while from ex. (2)

$$u = \frac{[12 \times 8 \times 6 \times 8 - 8 \times 8 \times 4(9\tfrac{1}{3})]62.5}{20000}.$$

That is, $u = 6.93$ feet, which locates C. Hence the pressure of the gate upon its hinges or other support is the same (aside from its own weight), provided it is rigid, as if the single horizontal force $R = 10$ tons acted at the point C, 2.93 ft. below the level of the tail-water surface.

443. If the plate, or gate, is entirely below the tail-water surface, the resultant pressure is applied in the centre of gravity of the plate.—Proof as follows: Conceive the surface to be divided into a great number of small equal areas, each $= dF$; then, the head of water of any dF being $= x_1$ on the head-water side, and $= x_2$ on the tail-water side, the resultant pressure on the dF is $\gamma dF(x_1 - x_2) = \gamma h dF$, in which h is the difference of level between head and tail water. That is, the resultant pressures on the equal dF's are *equal*, and hence form a system of equal parallel forces distributed over the plate in the same manner as the weights of the corresponding portions of the plate; therefore their single resultant acts through the centre of gravity of the plate; Q. E. D. This single resultant $= \int \gamma h dF = \gamma h \int dF = F h \gamma$.

EXAMPLE.—Fig. 494. The resultant pressure on a circular disk ab of radius $= 8$ inches, (in the vertical partition OK,) which has its centre of gravity 3 ft. below the tail-water surface, with $h = 2$ ft., is (ft., lb., sec.)

$$R = F h \gamma = \pi r^2 h \gamma$$

$$= \pi 8^2 \times 24 \times \frac{62.5}{1728} = 174.6 \text{ lbs.},$$

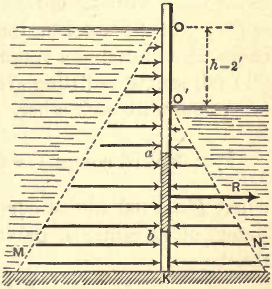

FIG. 494.

and is applied through the *centre of gravity* of the circle. *Evidently R is the same for any depth below the tail-water surface, so long as $h = 2$ ft.* [Let the student find a graphic proof of this statement.]

444. Liquid Pressure on Curved Surfaces.—If the rigid surface is curved, the pressures on the individual dF's, or elements of area, do not form a system of parallel forces, and the single resultant (if one is obtainable) is not equal to their sum. In general, the system is not equivalent to a single force, but can always be reduced to two forces (§ 38) the point of application of *one of which* is arbitrary (the arbitrary origin of § 38) and its amount $= \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$.

A single **Example** will be given; that of a thin rigid shell having the shape of the curved surface of a right cone, Fig. 495, its altitude being h and radius of base $= r$. It has no bottom, is placed on a smooth horizontal table, vertex up, and is filled with water through a small hole in the apex O, which is left open (to admit atmospheric pressure). What load, besides its own weight G', must be placed upon it to prevent the water from lifting it and escaping under the edge A? The pressure on each dF of the inner curved surface is $z\gamma dF$ and is normal to the surface. Its vertical compon. is $z\gamma dF \sin \alpha$, and horizontal compon. $= z\gamma dF \cos \alpha$. The dF's have all the same α, but different z's (or heads of water). The lifting tendency of the water on the thin shell is due to the vertical components forming a system of $\parallel$ forces, while the horizontal components, radiating symmetrically from the axis of the cone, neutralize each other. Hence the resultant lifting force is

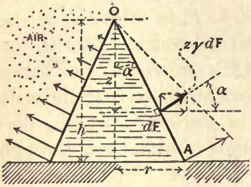

FIG. 495.

$$V = \Sigma(\text{vert. comps.}) = \gamma \sin \alpha \int z dF = \gamma \sin \alpha \, F \bar{z}; \quad (1)$$

where $F =$ total area of curved surface, and $\bar{z} =$ the "head of water" of its centre of gravity. Eq. (1) may also be written thus:

$$V = \gamma F_b \bar{z}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which $F_b = F \sin \alpha =$ area of the circular base $=$ area of the *projection of the curved surface upon a plane* $\urcorner$ to the vertical, i.e., upon a horizontal plane. Hence we may write

$$V = \tfrac{2}{3}\gamma \pi r^2 h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since $\bar{z} = \frac{2}{3}h$, being the z of the centre of gravity of the curved surface *and not that of the base.* $\gamma =$ heaviness of water. If $G' =$ weight of the shell and is $< V$, an additional load of $V - G'$ will be needed to prevent the lifting. If the shell has a bottom of weight $= G''$, forming a base for the cone and rigidly attached to it, we find that the vertical forces acting on the whole rigid body, base and all, are: V upward; G' and G'' downward; and the liquid pressure on the base, viz., $V' = \pi r^2 h\gamma$ (§ 428*a*) also downward. Hence the resultant vertical force to be counteracted by the table is downward, and

$$= G' + G'' + V' - V, \text{ which} = G' + G'' + \tfrac{1}{3}\pi r^2 h\gamma; \quad (4)$$

i.e., the total weight of the rigid vessel and the water in it, as we know, of course, in advance.

CHAPTER III.

EARTH PRESSURE AND RETAINING WALLS.

[NOTE.—This chapter was outlined and written mainly by Prof. C. L. Crandall, and is here incorporated with his permission. The theory of earth pressure is arranged from Baumeister.]

445. Angle of Repose.—Granular materials, like dry sand, loose earth, soil, gravel, pease, shot, etc., on account of the friction between the component grains, occupy an intermediate position between liquids and large rigid bodies. When heaped up, the side of the mass cannot be made to stand at an inclination with the horizontal greater than a definite angle called the *angle of natural slope*, or *angle of repose*, different for each material; so that if the side of the mass is to be retained permanently at some greater angle, a *Retaining Wall* (or "*Revetment Wall*," in military parlance) becomes necessary to support it. If the material is somewhat moist it may be made to stand alone at an inclination greater than that of the natural slope, on account of the cohesion thus produced, but only as long as the degree of moisture remains; while if much water is present, it assumes the consistency of mud and may require a much thicker wall, if it is to be supported laterally, than if dry.

In dealing with earth to be supported by a retaining wall, we consider the former to have lost any original cohesion which may have existed among its particles, or that it will eventually lose it through the action of the weather; and hence treat it as a granular material.

A few approximate values of the angle of natural slope are given below, being taken from Fanning, p. 345; see reference on p. 538 of this work.

MATERIAL.	Angle of Repose.	Coefficient of Friction.	Ratio of Slope.
			Horiz. to vert.
Dry sand, fine	28°	.532	1.88 to 1
" " coarse	30°	.577	1.73 " 1
Damp clay	45°	1.000	1.00 " 1
Wet clay	15°	.268	3.73 " 1
Clayey gravel	45°	1.000	1.00 " 1
Shingle	42°	.900	1.11 " 1
Gravel	38°	.781	1.28 " 1
Firm loam	36°	.727	1.38 " 1
Vegetable soil	35°	.700	1.43 " 1
Peat	20°	.364	2.75 " 1

The angle of repose, or natural slope, is also, evidently, the angle of friction between two masses of the same granular material.

446. Earth Pressure, and Wedge of Maximum Thrust.—Fig. 496. Let AB be a retaining wall, having a plane face AB in contact with a mass of earth ABD, both wall and earth being of indefinite extent ⅂ to the paper.

Let AD be the natural slope of the earth, making an angle β with the vertical (β is the complement of the angle of repose; see preceding table). Since AB, making an angle α with the vertical, is more nearly vertical than AD, the retaining wall is necessary, to keep the mass ABD in the position shown. The profile BCD may be of *any form* in this general discussion. Suppose the wall to be on the point of giving way; then the following motions are impending:

1st. Sliding is impending between some portion $ABC'A$ of the mass of earth and the remainder $C'AD$, the *surface of rupture* AC' (C' not shown in figure because not found yet, but lying somewhere on the profile $BCID$) being assumed plane, and making some angle ϕ' (to be determined) with the vertical. At this instant the resultant pressure N' of $AC'D$ on the plane AC' of the mass ABC' (a wedge) must *make an angle* $= \beta$ ($=$ comp. of angle of friction) with AC' on the upper side.

2d. A downward sliding of the mass ABC' along the back face AB of the wall. That is, the resultant pressure P' of the wall against the mass BAC' at this instant *makes an angle*

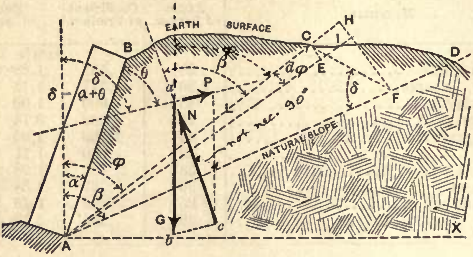

FIG. 496.

θ (= complement of angle of friction between the earth and wall) with the plane AB *and on the upper side.* The weight of the wedge of earth BAC' will be called G', and we desire to find the pressure P' against the wall.

Let BAC be a wedge (of the earth-mass), in which AC makes *any* angle ϕ with AV, and suppose *it* to be on the point of moving down and forcing out the wall; thus encountering friction both on the plane AC and the plane AB. Then the forces acting on it are three, acting in known directions; viz.: G, its own weight, vertical; N, the resultant pressure of the earth below it, making an angle β with AC on upper side; and P, the resultant pressure of the wall, at angle θ with AB (see Fig. 496 for positions of N and P). If now we express the force P in terms of ϕ and other quantities, and find that value ϕ', of ϕ, for which P is a maximum, we thereby determine the "*wedge of maximum thrust,*" $ABC'A$; while this maximum thrust, P', is the force which the wall must be designed to withstand. [If the wall is overturned, the earth will sink with it until this part of its surface gradually assumes the natural slope.]

Let G = weight of prism of base ABC, and altitude = unity ⏋ to paper; then $G = \gamma \times$ area ABC, where γ = "heaviness" = wgt. per cub. unit, of earth. Now P, G, and N balance; therefore, in triangle abc, if ab and ac are drawn ∥

and = G and N respectively, bc is = and $\parallel$ to P; and from Trigonometry we have

$$P = G \frac{\sin[\beta - \phi]}{\sin[\beta + \delta - \phi]}; \quad . \quad . \quad . \quad . \quad (1)$$

in which δ stands for $\alpha + \theta$, for brevity, being the angle which P makes with the vertical. N makes an angle $= \beta - \phi$ with the vertical.

The value, ϕ', of ϕ, which makes P a maximum is found by placing $\frac{dP}{d\phi} = 0$. From eq. (1), remembering that G is a function of ϕ, and that β and δ are constants, we have

$$\frac{dP}{d\phi} = \frac{\sin(\beta + \delta - \phi)\left[\frac{dG}{d\phi}\sin(\beta - \phi) - G\cos(\beta - \phi)\right] + G\sin(\beta - \phi)\cos(\beta + \delta - \phi)}{\sin^2[\beta + \delta - \phi]}.$$

For P to be a maximum we must put

$$\textit{numerator of above} = 0. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

To find a geometrical equivalent of $\frac{dG}{d\phi}$, denote $\overline{AC}$ by L, and draw AE, making an angle $= d\phi$ with AC. Now the area $ACI = \overline{AI} \times \frac{1}{2}\overline{CE} = (L + dL)\frac{1}{2}Ld\phi = \frac{1}{2}L^2 d\phi$ . . . (neglecting infinitesimal of 2d order). Now

$$dG = \gamma \times \text{ area } ACI \times \text{unity}; \therefore \frac{dG}{d\phi} = \tfrac{1}{2}\gamma L^2; \therefore (a) \text{ becomes}$$

$$\sin(\beta + \delta - \phi)\tfrac{1}{2}\gamma L^2 \sin(\beta - \phi) - \sin(\beta + \delta - \phi) G \cos(\beta - \phi)$$

$$+ G \sin(\beta - \phi)\cos(\beta + \delta - \phi) = 0;$$

i.e., $G =$

$$\frac{\frac{1}{2}\gamma L^2 \sin(\beta - \phi)\sin(\beta + \delta - \phi)}{\sin(\beta + \delta - \phi)\cos(\beta - \phi) - \cos(\beta + \delta - \phi)\sin(\beta - \phi)}$$

when P is a maximum; and hence, calling G' and ϕ' and L' the values of G, ϕ, and L, for max. P, we have

$$G' = \tfrac{1}{2}\gamma L'^2 \frac{\sin(\beta - \phi')\sin(\beta + \delta - \phi')}{\sin\delta}, \quad . \quad . \quad . \quad (2)$$

and therefore from (1) P max. itself is

$$P' = \tfrac{1}{2}\gamma L'^2 \,.\, \frac{\sin^2(\beta - \phi')}{\sin\delta}. \quad . \quad . \quad . \quad . \quad (3)$$

447. Geometric Interpretation and Construction.—If in Fig. 496 we draw CF, making angle δ with AD, C being any point on the ground surface BD, we have

$$\overline{CF} = L\frac{\sin(\beta - \phi)}{\sin\delta}.$$

Drop a perpendicular FH from F to AC, and we shall have

$$\overline{FH} = \overline{CF} \,.\, \sin(\beta + \delta - \phi), = L \,.\, \frac{\sin(\beta - \phi)\sin(\beta + \delta - \phi)}{\sin\delta}.$$

From this it follows that the weight of prism of base ACF and unit height

$$= \tfrac{1}{2}\gamma L \,.\, \overline{FH} = \tfrac{1}{2}\gamma L^2 \,.\, \frac{\sin(\beta - \phi)\sin(\beta + \delta - \phi)}{\sin\delta}. \quad (4)$$

When AC (as ϕ varies) assumes the position and value AC', bounding the prism of maximum thrust, Fig. 497, L becomes $= L'$, and $\phi = \phi'$; and eq. (4) gives the weight of the prism $AC'F'$. This weight is seen to be equal to that of the prism (or wedge) of maximum thrust ABC', by comparing eq. (4) with eq. (2); that is, *AC' bisects the area $ABC'F'$, and hence may be determined by fixing such a point C', on the upper profile BD, as to make the triangular area $AC'F'$ equal to the sectional area of the wedge $BC'A$*; $C'F'$ being drawn at an angle $= \delta$ with AD.

This holds for any form of ground surface BD, or any values of the constants β, α, or θ. C' is best found graphically by trial, in dealing with an irregular profile BD.

Having found AC', $= L'$, P' can be found from (3), or *graphically* as follows: (Fig. 497) With F' as a centre and radius $= C'F'$, describe an arc cutting AD in J', and join $C'J'$. The weight of prism with base $C'J'F'$ and unit height will $= P'$. For that prism has a weight

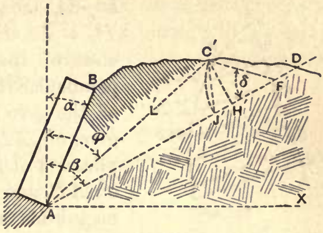

FIG. 497.

$$= \tfrac{1}{2}\gamma \cdot \overline{F'J'} \cdot \overline{C'H'};$$

but $$\overline{F'J'} = \overline{F'C'} = \frac{L' \sin(\beta - \phi')}{\sin\delta},$$

and $$\overline{C'H'} = L' \sin(\beta - \phi');$$

$$\therefore \quad \text{weight of prism } C'J'F = \tfrac{1}{2}\gamma L'^2 \frac{\sin^2(\beta - \phi')}{\sin\delta}; \; = P'.$$

[See eq. (3).]

448. Point of Application of the Resultant Earth Thrust.—This thrust (called P' throughout this chapter *except in the present paragraph*) is now known in magnitude and direction, but not in position; i.e.. we must still determine its line of action, as follows:

Divide AB into a number of equal parts, ab, bc, cd, etc.; see Fig. 498. Treat ab as a small retaining wall, and find the magnitude P' of the thrust against it by § 447; treat ac similarly, thus finding the thrust, P'', against it; then ad, ae, etc., the thrusts against them being found to be P''', P^{IV}, etc.; and so on. Now the pressure

P' on ab is applied nearly at middle of ab,
$P'' - P'$ " " " " bc,
$P''' - P''$ " " " " cd,

and so on. Erect perpendiculars at the middle points of *ab*, *bc*, *cd*, etc., equal respectively to P', $P'' - P'$, $P''' - P''$, etc., and join the ends of the perpendiculars. The perpendicular through the centre of gravity of the area so formed (Fig. 498) will give, on AB, the required point of application of the thrust or earth pressure on AB, and this, with the direction and magnitude already found in § 447, will completely determine the thrust against the wall AB.

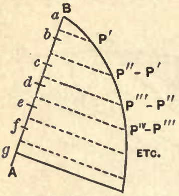

FIG. 498.

449. Special Law of Loading.—If the material to be retained consists of loose stone, masses of masonry, buildings, or even moving loads, as in the case of a wharf or roadway, each can be replaced by the same *weight* of earth or other material which will render the bank homogeneous, situated on the *same verticals*, and the profile thus reduced can be treated by §§ 447 and 448.

Should the solid mass extend below the plane of rupture, AC', and the plane of natural slope, it will become a retaining wall for the material beyond, if strong enough to act as such (limiting the profile $ABCD$ of Fig. 496 to the front of the mass, or to the front and line of rupture for maximum thrust above it, if it does not reach the surface); if not strong enough, or if it does not reach below the plane of natural slope, its presence is better ignored, probably, except that the increased weight must be considered.

The spandrel wall of an arch may present two of these special cases; i.e., the profile may be enlarged to include a moving load, while it may be limited at the back by the other spandrel.

If the earth profile starts at the front edge of the top of wall, instead of from the back as at B, Fig. 496, eq. (3) would only apply to the portion behind AB prolonged, leaving the part on the wall (top) to be treated as a part of the wall to aid in resisting the thrust.

If the wall is stepped in from the footings, or foundation courses, probably the weak section will be just above them; if stepped at intervals up the back of the wall, the surface of separation between the wall and filling, if it is plane, will probably pass through the first step and incline forward as much as possible without cutting the wall.

450. Straight Earth-profile.—The general case can be simplified as follows (the earth-profile BD being straight, at angle $= \zeta$ with vertical, $= DET$): Since the triangles ABC' and $C'AF'$ are equal, from § 447, and AC' is common, therefore $BS = F'H$ (both being drawn ⏋ to AC'). Draw AE and $BM \parallel$ to $F'C'$ (i.e., at angle δ with AD), cutting DB, prolonged, in E. We have

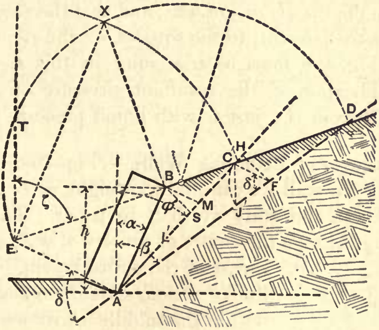

FIG. 499.

$$\frac{\overline{DE}}{\overline{C'E}} = \frac{\overline{EA}}{\overline{EA} - \overline{C'F'}}, \text{ and } \frac{\overline{C'E}}{\overline{BE}} = \frac{\overline{EA}}{\overline{EA} - \overline{BM}}.$$

But $C'F' = BM$ (since $\overline{BS} = \overline{H'F'}$);

therefore $$\frac{\overline{DE}}{\overline{C'E}} = \frac{\overline{C'E}}{\overline{BE}}; \text{ i.e., } \overline{DE} \,.\, \overline{BE} = \overline{C'E}^2,$$

which justifies the following construction for locating the desired point C' on BD, and thus finding $\overline{AC'} = L'$ and the angle ϕ': Describe a circle on ED as a diameter, and draw

BX ⅂ to BD, thus fixing X in the curve. With centre E describe a circular arc through X, cutting BD in C', required.

Having $\overline{AC'}$ (i.e., L'), ϕ' is known; hence from eq. (3) we obtain the earth thrust or pressure P': or, with F' as centre and radius $= C'F'$, describe arc $C'J'$; then the triangle $C'F'J'$ is the base of a prism of unity height whose weight $= P'$ (as in § 447).

Centre of Pressure.—Applying the method of § 448, Fig. 498, to this case, we find that the successive L''s are proportional to the depths ab, ac, ad, etc., and that the successive P's are proportional [see (3)] to the squares of the depths; hence the area in Fig. 498 must be triangular in this case, and the point of application of the resultant pressure on AB is *one third* of AB from A: just as with liquid pressure.

451. Resistance of Retaining Walls.—(Fig. 500.) Knowing the height of the wall we can find its weight, $= G_1$, for an assumed thickness, and unity width ⅂ to paper. The resultant of G_1, acting through the centre of gravity of wall, and P', the thrust of the embankment, in its proper line of action, should cut the base AV within the middle third and make an angle with the normal (to the base) less than the angle of friction.

For the straight wall and straight earth-profile of Fig. 499 and § 450, the length L', $= AC'$, can be expressed in terms of the (vertical) height, h, of wall, thus:

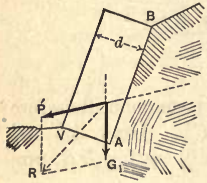

Fig. 500.

$$\overline{AB} = \frac{h}{\cos\alpha},$$

$$\text{and } L' = \overline{AC'} = \overline{AB}\,.\,\frac{\sin(\zeta-\alpha)}{\sin(\zeta-\phi')} = \frac{h}{\cos\alpha}\,.\,\frac{\sin(\zeta-\alpha)}{\sin(\zeta-\phi')};$$

∴ eq. (3) becomes

$$P' = \tfrac{1}{2}\gamma\,\frac{h^2}{\cos^2\alpha}\,.\,\frac{\sin^2(\beta-\phi')\sin^2(\zeta-\alpha)}{\sin\delta\,\sin^2(\zeta-\phi')} = \tfrac{1}{2}\gamma\,\frac{h^2}{\cos^2\alpha}\,.\,A. \quad (5)$$

[A representing the large fraction for brevity.]

This equation will require, for a wall of rectangular section, that the thickness, d, increase as h, in order that its weight may increase as h^2 (i.e., as P') and that its resisting moment may increase with the overturning moment.

By this equality of moments is meant that $P'a = G_1b$; where a and b are the respective lever-arms of the two forces about the front edge of the middle third. (AB is the back of the wall.) In other words, their resultant will pass through this point.

The following table is computed on the basis just mentioned, viz., *that the resultant of P' and G shall pass through the front edge of the midde third.*

The symbols of eq. (5) and the table are all shown in Fig. 499, except γ, θ, and δ. $\gamma =$ weight of a cubic foot of earth, here assumed $= \frac{2}{3}$ that of masonry (e.g., if earth weighs 100 lbs., masonry is assumed to weigh 150 lbs. per cubic foot); $\theta =$ angle which the thrust P' makes with the back of the wall; and $\delta = \alpha + \theta, = \theta$ in this case as *the wall is vertical*, or $\alpha = 0$. d is the proper safe thickness to be given to the wall, of rectangular section, to prevent overturning, as stated above; h is the altitude, and A is the fraction shown in eq. (5).

Whether the wall is safe against sliding on its base, and whether a safe compression per unit area is exceeded on the front edge of the base, are matters for separate consideration. The latter will seldom govern with ordinary retaining walls.

$\alpha = 0$; i.e., wall is vertical; also density of wall $= \frac{3}{2}$ that of the earth.

		I.			II.			III.		
		$\zeta = 90°$ $\theta = 90°$			$\zeta = 90°$ $\theta = \beta$			$\zeta = \beta$ $\theta = \beta$		
$\tan\beta$	β	ϕ'	A	d	ϕ'	A	d	ϕ'	A	d
1.0	45°	22½°	.17	.34h	26°	.18	.22h	45°	.71	.33h
1.5	56⅓°	28°	.29	.44h	33°	.26	.30h	56°	.83	.46h
2.0	63½°	31¾°	.38	.51h	38°	.33	.36h	63°	.89	.51h
4 0	76°	38°	.61	.64h	45°	.54	.50h	76°	.97	.65h
Infinity	90°	45°	1.00	.82h	90°	1.00	.82h	90°	1.00	.82h

In Case I of table, since $\alpha = 0$, $\theta = 90°$ and $\zeta = 90°$; $\therefore\ \delta = 90°$, and hence $C'F''$ of Fig. 499 is $\urcorner$ to AD, so that (since the area of $\triangle ABC' = \triangle AC'F'$) ϕ' must $= \frac{1}{2}\beta$. These values, in (5), give

$$P' = \tfrac{1}{2}\gamma h^2 \tan^2 \tfrac{1}{2}\beta; \text{ i.e., } A = \tan^2 \tfrac{1}{2}\beta. \quad . \quad . \quad . \quad (6)$$

In Case II, since $\zeta = 90°$, $\alpha = 0$ and $\theta = \beta$, $\therefore\ \delta = \beta$; and (5) reduces to

$$P' = \tfrac{1}{2}\gamma h^2 \frac{\sin^2(\beta - \phi')}{\sin\beta\cos^2\phi'}; \text{ i.e., } A = \frac{\sin^2(\beta - \phi')}{\sin\beta\cos^2\phi'}. \quad . \quad . \quad (7)$$

In Case III, $\zeta = \beta$ and BD will be $\parallel$ to AD, D being at infinity. See Fig. 501. Through B draw $BH \urcorner$ to AD, and BF'' making angle δ with AD. C' is now to be located on BD, so as to make (area of) $\triangle ABC' =$ (area of) $\triangle AC'F'$ (according to § 447), the angle $C'F'A$ being $= \delta = \alpha + \theta$; $= \theta$, in this case, and hence also $= \beta$. Conceive B and F' to be joined.

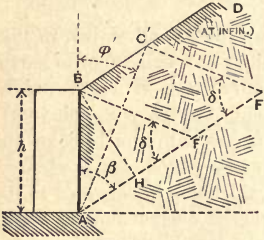

FIG. 501.

Now $\triangle AC'F' = \triangle ABF'' + \triangle BF'F''$.

But $\triangle ABC' = \triangle BF'F''$ (equal bases and altitudes).

Hence $\triangle ABC'$ *cannot* $=$ $\triangle AC'F'$ *unless* C' *is moved out to infinity;* and then ϕ' becomes $= \beta$, and eq. (5) reduces to

$$P' = \tfrac{1}{2}\gamma h^2 \sin\beta; \text{ i.e., } A = \sin\beta. \quad . \quad . \quad . \quad (8)$$

[Increasing α from zero will decrease the thickness d; i.e., inclining the wall inwards will decrease the required thickness, but diminish the frictional stability at the base, unless the latter be $\urcorner$ to AB. The back of the wall is frequently inclined outwards, making the section a trapezoid, to increase the frictional stability at the base when necessary, as with timber walls supporting water.]

452. Practical Considerations. — An examination of the values of A and d in the table of § 451 will show that in supporting quicksand and many kinds of clay which are almost fluid under the influence of water, it is important to know what kind of drainage can be secured, for on that will depend the thickness of the wall. With well compacted material free from water-bearing strata, an assumed natural slope of 1½ to 1 (i.e., 1½ hor. to 1 vert.) will be safe; the actual pressure below the effect of frost and surface water will be that due to a much steeper slope on account of *cohesion* (neglected in this theory).

The thrust from freshly placed material can be reduced by depositing it in layers sloping back from the wall. If it is not so placed, however, the natural slope will seldom be flatter than 1½ to 1 unless reduced by water. In supporting material which contains water-bearing strata sloping toward the wall and overlain by strata which are liable to become semi-fluid and slippery, the thrust may exceed that due to semi-fluid material on account of the surcharge. If these strata are under the wall and cannot be reached by the foundation, or if resistance to sliding cannot be obtained from the material in front by sheet-piling, no amount of masonry can give security.

Water at the back of the wall will, by freezing, cause the material to exert an indefinitely great pressure, besides disintegrating the wall itself. If there is danger of its accumulation, drainage should be provided by a layer of loose stone at the back leading to "weep-holes" through the wall.

A friction-angle at the back of the wall equal to that of the filling should always be realized by making the back rough by steps, or projecting stones or bricks. Its effect on the required thickness is too great to be economically ignored.

The resistance to slipping at the base can be increased, when necessary, by inclining the foundation inwards; by stepping or sloping the back of the wall so as to add to its effective weight or incline the thrust more nearly to the vertical; by sheet-piling in front of the foundation, thus gaining the resistance offered by the piles to lateral motion; by deeper foundations, gaining the resistance of the earth in front of the wall.

The coefficient of friction on the base ranges, according to Trautwine, from 0.20 to 0.30 on wet clay;
" .50 to .66 " dry earth;
" .66 to .75 " sand or gravel;
" .60 on a dry wooden platform; to .75 on a wet one.

If the wall is partially submerged, the buoyant effort should be subtracted from G_1, the weight of wall.

453. Results of Experience.—(Trautwine.) In railroad practice, a vertical wall of rectangular section, sustaining sand, gravel, or earth, level with the top [p. 682 of Civ. Eng. Pocket Book] and loosely deposited, as when dumped from carts, cars, etc., should have a thickness d, as follows:

If of cut stone, or of first-class large ranged rubble, in mortar. . . $d = .35h$
" good common scabbled mortar-rubble, or brick. . . . . $d = .40h$
" well scabbled dry rubble. . . . . $d = .50h$

Where h includes the total height, or about 3 ft. of foundations.

(a) For the best masonry of its class h may be taken from the top of the foundation in front.

(b) A mixture of sand or earth, with a large proportion of round boulders or cobbles, will weigh more than the backing assumed above; requiring d to be increased from one eighth to one sixth part.

(c) The wall will be stronger by inclining the back inwards, especially if of dry masonry, or if the backing is put in place before the mortar has set.

(d) The back of the wall should be left rough to increase friction.

(e) Where deep freezing occurs, the back should slope outward for 3 or 4 feet below the top and be left smooth.

(f) When a wall is too thin, it will generally fail by bulging outward at about one third the height. The failure is usually gradual and may take years.

(g) *Counterforts*, or buttresses at the back of the wall, usually of rectangular section, may be regarded as a waste of masonry, although considerably used in Europe; the bond will seldom hold them to the wall. Buttresses *in front* add to the strength, but are not common, on account of expense.

(*h*) *Land-ties* of iron or wood, tying the wall to anchors imbedded below the line of natural slope, are sometimes used to increase stability.

(*i*) Walls with curved cross-sections are not recommended.

454. Conclusions of Mr. B. Baker.—("Actual Lateral Pressure of Earthwork.") Experience has shown that $d = 0.25h$, with batter of 1 to 2 inches per foot on face, is sufficient when backing and foundation are both favorable; also that under *no* ordinary conditions of surcharge or heavy backing, with solid foundation, is it necessary for d to be greater than $0.50h$.

Mr. Baker's own rule is to make $d = 0.33h$ at the top of the footings, with a face batter of 1½ inches per foot, in ground of average character; and, if any material is taken out to form a face-panel, three fourths of it is put back in the form of a pilaster. The object of the batter, and of the panel if used, is to distribute the pressure better on the foundation. All the walls of the "District Railway" (London) were designed on this basis, and there has not been a single instance of settlement, of overturning, or of sliding forward.

455. Experiments with Models.—Accounts of experiments with apparatus on a small scale, with sand, etc., may be found in vol. LXXI of Proceedings of Institution of Civil Engineers, London, England (p. 350); also in vol. II of the "Annales des Ponts et Chaussées" for 1885 (p. 788).

The results of these experiments, and the results of experience given in §§ 453 and 454, when compared with the table of p. 581, indicate a fairly close agreement between practice and theory. This agreement is believed to be close enough so that the general method of §§ 447 and 451, with the table of p. 581, can be relied upon in practice. The greatest value of this method will, of course, be for cases of exceptional loading, inclined walls, etc., where the results of experience do not furnish so valuable a guide.

CHAPTER IV.

HYDROSTATICS (*Continued*)—IMMERSION AND FLOTATION.

456. Rigid Body Immersed in a Liquid. Buoyant Effort.—If any portion of a body of homogeneous liquid at rest be conceived to become rigid without alteration of shape or bulk, it would evidently still remain at rest; i.e., its weight, applied at its centre of gravity, would be balanced by the pressures, on its bounding surfaces, of the contiguous portions of the liquid; hence,

If a rigid body or solid is immersed in a liquid, both being at rest, the resultant action upon it of the surrounding liquid (or fluid) is a vertical upward force called the "buoyant effort," equal in amount to the weight of liquid displaced, and acting through the centre of gravity of the volume (considered as homogeneous) of displacement (now occupied by the solid). This point is called *the centre of buoyancy,* and is sometimes spoken of as *the centre of gravity of the displaced water.* If V' = the volume of displacement, and γ = heaviness of the liquid, then the

$$\textit{buoyant effort} = V'\gamma. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(By "volume of displacement" is meant, of course, the volume of liquid actually displaced when the body is immersed.)

If the weight G' of the solid is not equal to the buoyant effort, or if its centre of gravity does not lie in the same vertical as the centre of buoyancy, the two forces form an unbalanced system and motion begins. But as a consequence of this very motion the action of the liquid is modified in a manner dependent on the shape and kind of motion of the body.

Problems in this chapter are restricted to cases of rest, i.e., balanced forces.

Suppose $G' = V'\gamma$; then,

If the centre of gravity lies in the same vertical line as the centre of buoyancy and *underneath* the latter, the equilibrium is *stable;* i.e., after a slight angular disturbance the body returns to its original position (after several oscillations); while if *above* the latter, the equilibrium is *unstable.* If they *coincide*, as when the solid is homogeneous (but not hollow), and of the same heaviness (§ 7) as the liquid, the equilibrium is *indifferent*, i.e., possible in any position of the body.

The following is interesting in this connection:

In an account of the new British submarine boat "Nautilus," a writer in *Chambers's Journal* remarked [1887]: "At each side of the vessel are four port-holes, into which fit cylinders two feet in diameter. When these cylinders are projected outwards, as they can be by suitable gearing, the displacement of the boat is so much increased that the vessel rises to the surface; but when the cylinders are withdrawn into their sockets, it will sink."

As another case in point, large water-tight canvas "air-bags" have recently been used for raising sunken ships. They are sunk in a collapsed state, attached by divers to the submerged vessel, and then inflated with air from pumps above, which of course largely augments their displacement while adding no appreciable weight.

457. Examples of Immersion.—Fig. 502. At (*a*) is an example of stable equilibrium, the centre of buoyancy B being above the centre of gravity C, and the buoyant effort $V'\gamma = G' =$ the weight of the solid; at (a'), conversely, we have unstable equilibrium, with $V'\gamma$ still $= G'$. At (*b*) the buoyant effort $V'\gamma$ is $> G'$, and

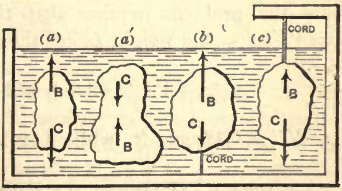

FIG. 502.

to preserve equilibrium the body is attached by a cord to the bottom of the vessel. The tension in this cord is

$$S_b = V'\gamma - G'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At (c) $V'\gamma$ is $< G'$, and the cord must be attached to a support above, and its tension is

$$S_c = G' - V'\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If in eq. (2) [(c) in figure] we call S_c the *apparent weight* of the immersed body, and measure it by a spring- or beam-balance, we may say that

The apparent weight of a solid totally immersed in a liquid equals its real weight diminished by that of the amount of liquid displaced; in other words, *the loss of weight = the weight of displaced liquid.*

Example 1.—How great a mass (not hollow) of cast-iron can be supported in water by a wrought-iron cylinder weighing 140 lbs., if the latter contains a vacuous space and displaces 3 cub. feet of water, both bodies being completely immersed? [Ft., lb., sec.]

The buoyant effort on the cylinder is

$$V'\gamma = 3 \times 62.5 = 187.5 \text{ lbs.,}$$

leaving a residue of 47.5 lbs. upward force to buoy the cast-iron, whose volume V'' is unknown, while its heaviness (§ 7) is $\gamma'' = 450$ lbs. per cub. foot. The direct buoyant effort of the water on the cast-iron is $V''\gamma = [V'' \times 62.5]$ lbs., and the problem requires that this force $+ 47.5$ lbs. shall $= V''\gamma'' =$ the weight G'' of the cast-iron;

$$\therefore \quad V'' \times 62.5 + 47.5 = V'' \times 450;$$

$\therefore \; V'' = 0.12$ cub. ft., while $0.12 \times 450 = 54$ lbs. of cast-iron. *Ans.*

Example 2.—Required the volume V', and heaviness γ', of a *homogeneous* solid which weighs 6 lbs. out of water and 4 lbs. when immersed (*apparent weight*) (ft., lb., sec.).

From eq. (2), $4 = 6 - V' \times 62.5$; $\therefore V' = 0.032$ cub. feet;

$$\therefore \gamma' = G' \div V' = 6 \div 0.032 = 187.5 \text{ lbs. per cub. ft.},$$

and the ratio of γ' to γ is $187.5 : 62.5 = 3.0$ (abstract number); i.e., the substance of this solid is three times as dense, or three times as heavy, as water. [The buoyant effort of the air has been neglected in giving the true weight as 6 lbs.]

458. Specific Gravity.—By *specific gravity* is meant the ratio of the heaviness of a given homogeneous substance to that of a standard homogeneous substance; in other words, the ratio of the weight of a certain volume of the substance to the weight of an *equal* volume of the standard substance. Distilled water at the temperature of maximum density (4° Centigrade) under a pressure of 14.7 lbs. per sq. inch is sometimes taken as the standard substance, more frequently, however, at 62° Fahrenheit (16°.6 Centigrade). Water, then, being the standard substance, the numerical example last given illustrates a common method of determining experimentally the specific gravity of a homogeneous solid substance, the value there obtained being 3. The symbol σ will be used to denote specific gravity, which is evidently an abstract number. The standard substance should always be mentioned, and its heaviness γ; then the heaviness of a substance whose specific gravity is σ is

$$\gamma' = \sigma\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the weight G' of any volume V' of the substance may be written

$$G' = V'\gamma' = V'\sigma\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Evidently a knowledge of the value of γ' dispenses with the use of σ, though when the latter can be introduced into problems involving the buoyant effort of a liquid the criterion as to whether a *homogeneous* solid will sink or rise, when immersed in the *standard* liquid, is more easily applied, thus: Being immersed, the volume V' of the body = that, V, of displaced liquid. Hence, if G' is $> V'\gamma$, i.e., if $V'\gamma'$ is $> V'\gamma$, or $\sigma > 1$, it sinks;

while if G' is $< V'\gamma$, ... or $\sigma < 1$, it rises;

i.e., according as the weight G' is $>$ or $<$ than the buoyant effort.

Other methods of determining the specific gravity of solids, liquids, and gases are given in works on Physics.

459. Equilibrium of Flotation.—In case the weight G' of an immersed solid is less than the buoyant effort $V'\gamma$ (where V' is the volume of displacement, and γ the heaviness of liquid) the body rises to the surface, and after a series of oscillations comes to rest in such a position, Fig. 503, that its centre of gravity C and the centre of buoyancy B (the new B, belonging to the new volume of displacement, which is limited above by the horizontal plane of the free surface of the liquid) are in the same vertical (called the axis of flotation, or line of support), and that the volume of displacement has diminished to such a new value V, that

$$V\gamma = G'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the figure, $V =$ vol. AND, below the horizontal plane AN, and the slightest motion of the body *will change the form of this volume*, in general (whereas with complete immersion the volume of displacement remains constant). For stable equilibrium it is not essential in every case that C (centre of gravity of body) should be below B (the centre of buoyancy) as with complete immersion, since if the solid is turned, B may change its position in the body, as the form of the volume AND changes.

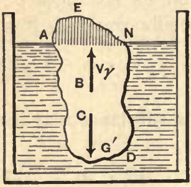

FIG. 503.

There is now no definite relation between the volume of displacement V and that of the body, V', unless the latter is *homogeneous*, and then for G' we may write $V'\gamma'$, i.e.

$$V'\gamma' = V\gamma \text{ (for a homogeneous solid)}; \quad . \quad . \quad (2)$$

or, *the volumes are inversely proportional to the heavinesses.*

The buoyant effort of the air on the portion ANE may be neglected in most practical cases, as being insignificant.

If the solid is *hollow*, the position of its centre of gravity C may be easily varied (by shifting ballast, e.g.) within certain limits, but that of the centre of buoyancy B depends only on the geometrical form of the volume of displacement AND, below the horizontal plane AN.

EXAMPLE.—(Ft., lb., sec.) Will a solid weighing $G' = 400$ lbs., and having a volume $V' = 8$ cub. feet, without hollows or recesses, float in water? To obtain a buoyant effort of 400 lbs., we need a volume of displacement, see eq. (1), of

$$V = \frac{G'}{\gamma} = \frac{400}{62.5} = \text{only } 6.4 \text{ cub. ft.}$$

Hence the solid will float with $8 - 6.4$, or 1.6, cub. ft. projecting above the water level.

Query: A vessel contains water, reaching to its brim, and also a piece of ice which floats without touching the vessel. When the ice melts will the water overflow?

460. The Hydrometer is a floating instrument for determining the relative heavinesses of liquids. Fig. 504 shows a simple form, consisting of a bulb and a cylindrical stem of glass, so designed and weighted as to float upright in all liquids whose heavinesses it is to compare. Let F denote the uniform sectional area of the stem (a circle), and suppose that when floating in water (whose heaviness $= \gamma$) the water surface marks a point A on the stem; and that when floating in another liquid, say petroleum, whose heaviness, $= \gamma_p$, we wish to determine, it floats at a greater depth, the liquid surface now marking A' on the stem, a height $= x$ above A. G' is the same in both experiments; but while the volume of displacement in water is V, in petroleum it is $V + Fx$. Therefore from eq. (1), § 459,

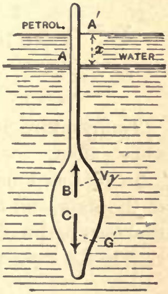

FIG. 504.

$$\textit{in the water} \qquad G' = V\gamma, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\textit{in the petroleum} \quad G' = (V + Fx)\gamma_p; \quad . \quad . \quad (2)$$

from which, knowing G', F, x, and γ, we find V and γ_p, i.e.,

$$V = \frac{G'}{\gamma} \quad \text{and} \quad \gamma_p = \frac{G'\gamma}{G' + Fx\gamma}. \quad . \quad . \quad . \quad (3)$$

[N.B.—F is best determined by noting the additional distance, $= l$, through which the instrument sinks in water under an additional load P, *not immersed;* for then

$$G' + P = (V + Fl)\gamma, \quad \text{or} \quad F = \frac{P}{l\gamma}.]$$

EXAMPLE.—[Using the *inch*, *ounce*, and *second*, in which system $\gamma = 1000 \div 1728 = 0.578$ (§ 409).] With $G' = 3$ ounces, and $F = 0.10$ sq. inch, x being observed, on the graduated stem, to be 5 inches, we have for the petroleum

$$\gamma_p = \frac{3 \times 0.578}{3 + 0.10 \times 5 \times 0.578} = 0.525 \text{ oz. per cubic inch}$$

$$= 56.7 \text{ lbs. per cub. foot.}$$

Temperature influences the heaviness of most liquids to some extent.

In another kind of instrument a scale-pan is fixed to the top of the stem, and the specific gravity computed from the weight necessary to be placed on this pan to cause the hydrometer to sink to the *same* point in *all* liquids for which it is used.

461. Depth of Flotation.—If the weight and external shape of the floating body are known, and the centre of gravity so situated that the position of flotation is known, the *depth of the lowest point below the surface may be determined.*

CASE I. *Right prism or cylinder with its axis vertical.*—Fig. 505. (For stability in this position, see § 464*a*.) Let G' = weight of cylinder, F the area of its cross-section (full circle), h' its altitude, and h the unknown depth of flotation (or *draught*); then from eq. (1), § 426,

$$G' = Fh\gamma; \therefore h = \frac{G'}{F\gamma}; \quad (1)$$

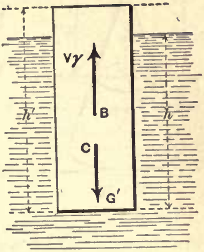

FIG. 505.

in which γ = heaviness of the liquid. If the prism (or cylinder) is *homogeneous* (and then C, at the middle of h', is higher than B) and γ' its heaviness, we then have

$$h = \frac{Fh'\gamma'}{F\gamma} = \frac{\gamma' h'}{\gamma} = \sigma h'; \quad . \quad . \quad . \quad . \quad (2)$$

in which σ = specific gravity of solid referred to the liquid as standard. (See § 458.)

CASE II. *Pyramid or cone with axis vertical and vertex down.*—Fig. 506. Let V' = volume of whole pyramid (or cone), and V = volume of displacement. From similar pyramids,

$$\frac{V}{V'} = \frac{h^3}{h'^3}; \therefore h = \sqrt[3]{\frac{V}{V'}} \cdot h'.$$

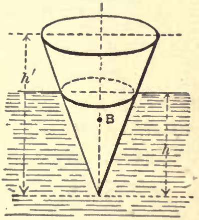

FIG. 506.

But $G' = V\gamma$; or, $V = \frac{G'}{\gamma}$; whence

$$h = h' \sqrt[3]{\frac{G'}{V'\gamma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Case III. *Ditto, but vertex up.*—Fig. 507. Let the notation be as before, for V and V'. The part out of water is a pyramid of volume $= V'' = V' - V$, and is similar to the whole pyramid;

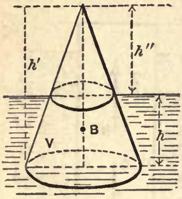

Fig. 507.

$$\therefore\ V' - V : V' :: h''^3 : h'^3.$$

Also, $$V = \frac{G'}{\gamma};$$

$$\therefore\ h'' = h'\sqrt[3]{\frac{V' - V}{V'}} = h'\sqrt[3]{\frac{V'\gamma - G'}{V'\gamma}};$$

$$\therefore\text{, finally, } h = h'\left[1 - \sqrt[3]{1 - [G' \div V'\gamma]}\right]. \quad . \quad . \quad . \quad (4)$$

Case IV. *Sphere.*—Fig. 508. The volume immersed is

$$V = \int_{z=0}^{z=h} (\pi x^2)dz = \pi\int_0^h (2rz - z^2)dz = \pi h^2\left[r - \frac{h}{3}\right];$$

and hence, since $V\gamma = G' =$ weight of sphere,

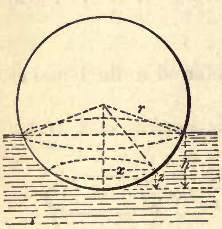

Fig. 508.

$$\pi r h^2 - \frac{\pi h^3}{3} = \frac{G'}{\gamma}. \quad . \quad . \quad (5)$$

From which cubic equation h may be obtained by successive trials and approximations.

[An exact solution of (5) for the unknown h is impossible, as it falls under the irreducible case of Cardan's Rule.]

Case V. *Right cylinder with axis horizontal.*—Fig. 509.

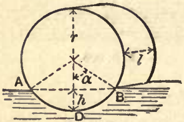

Fig. 509.

$$\left.\begin{array}{l}\text{Vol. im-}\\ \text{mers.} = V\end{array}\right\} = [\text{area of seg. } ADB] \times l$$

$$= (r^2\alpha - \tfrac{1}{2}r^2 \sin 2\alpha)l;$$

hence, since $V = \dfrac{G'}{\gamma}$,

$$lr^2[\alpha - \tfrac{1}{2}\sin 2\alpha] = \frac{G'}{\gamma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From this transcendental equation we can obtain α, by trial, in *radians* (see example in § 428), and finally h, since

$$h = r(1 - \cos \alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

EXAMPLE 1.—A sphere of 40 inches diameter is observed to have a depth of flotation $h = 9$ in. in water. Required its weight G'. From eq. (5) (inch, lb., sec.) we have

$$G' = [62.5 \div 1728]\pi 9^2[20 - \tfrac{1}{3} \times 9] = 156.5 \text{ lbs.}$$

The sphere may be hollow, e.g., of sheet metal loaded with shot; constructed in any way, so long as G' and the volume V of displacement remain unchanged. But if the sphere is *homogeneous*, *its heaviness* (§ 7) γ' must be

$$= G' \div V' = G' \div \tfrac{4}{3}\pi r^3 = (156.5) \div \tfrac{4}{3}\pi 20^3$$
$$= .00466 \text{ lbs. per cubic inch,}$$

and hence, referred to water, its specific gravity is $\sigma =$ about 0.13.

EXAMPLE 2.—The right cylinder in Fig. 509 is homogeneous and 10 inches in diameter, and has a specific gravity (referred to water) of $\sigma = 0.30$. Required the depth of flotation h.

Its heaviness must be $\gamma' = \sigma\gamma$; hence its weight

$$G' = V'\sigma\gamma = \pi r^2 l\sigma\gamma;$$

hence, from eq. (6),

$$r^2 l[\alpha - \tfrac{1}{2} \sin 2\alpha] = \pi r^2 l\sigma, \ \therefore \ \alpha - \tfrac{1}{2} \sin 2\alpha = \pi\sigma$$

(involving abstract numbers only). Trying $\alpha = 60°$ ($= \frac{1}{3}\pi$ in *radians*), we have

$$\tfrac{1}{3}\pi - \tfrac{1}{2} \sin 120° = 0.614; \text{ whereas } \pi\sigma = .9424$$

For $\alpha = 70°$, $1.2217 - \frac{1}{2} \sin 140° = 0.9003$;
For $\alpha = 71°$, $1.2391 - \frac{1}{2} \sin 142° = 0.9313$;
For $\alpha = 71° \, 22'$, $1.2455 - \frac{1}{2} \sin 142° \, 44' = 0.9428$, which may be considered sufficiently close. Now from eq. (7),

$$h = (5 \text{ in.})\,(1 - \cos 71° \, 22') = 3.40 \text{ in.}—\textit{Ans.}$$

462. Draught of Ships.—In designing a ship, especially if of a new model, the position of the centre of gravity is found by eq. (3) of § 23 (with weights instead of volumes); i.e., the sum of the products obtained by multiplying the weight of each portion of the hull and cargo by the distance of its centre of gravity from a convenient reference-plane (e.g., the horizontal plane of the keel bottom) is divided by the sum of the weights, and the quotient is the distance of the centre of gravity of the whole from the reference-plane.

Similarly, the distance from another reference-plane is determined. These two co-ordinates and the fact that the centre of gravity lies in the median vertical plane of symmetry of the ship (assuming a symmetrical arrangement of the framework and cargo) fix its location. The total weight, G', equals, of course, the sum of the individual weights just mentioned. The *centre of buoyancy*, for *any assumed draught* and corresponding position of ship, is found by the same method; but more simply, since it is the centre of gravity of the imaginary homogeneous volume between the water-line plane and the *wetted* surface of the hull. This volume (of "displacement") is divided into an even number (say 4 to 8) of horizontal laminæ of *equal thickness*, and Simpson's Rule applied to find the volume (i.e., the V of preceding formulæ), and also (eq. 3, § 23) the height of its centre of gravity above the keel. Similarly, by division into (from 8 to 20) vertical slices, $\urcorner$ to keel (an even number and of *equal thickness*), we find the distance of the centre of gravity from the bow. Thus the centre of buoyancy is fixed, and the corresponding buoyant effort $V\gamma$ (technically called the *displacement* and usually expressed in tons) computed, for any assumed draught of ship (upright). That position in which the "displacement" $= G' =$ weight of ship is the position of equilibrium of the ship when floating upright in still water, and the corresponding draught is noted. As to whether this equilibrium is stable or unstable, the following will show.

In most ships the centre of gravity C is several feet above the centre of buoyancy, B, and a foot or more below the water line.

After a ship is afloat and its draught actually noted its total weight G', $= V\gamma$, can be computed, the values of V for different draughts having been calculated in advance. In this way the weights of different cargoes can also be measured.

EXAMPLE.—A ship having a displacement of 5000 tons is itself 5000 tons in weight, and displaces a volume of *salt* water $V = G' \div \gamma = 10,000,000$ lbs. $\div$ 64 lbs. per cub. ft. $= 156250$ cub. ft.

463. Angular Stability of Ships.—If a vessel floating upright were of the peculiar form and position of Fig. 510 (the water-line section having an area = zero) its tendency to regain that position, or depart from it, when slightly inclined an angle ϕ from the vertical is due to the action of the couple now formed by the equal and parallel forces $V\gamma$ and G', which are no longer directly opposed. This couple is called a *righting couple* if it acts to restore the first position (as in Fig. 511, where C is lower than B), and an upsetting couple if the reverse, C above B. In either case the moment of the couple is

$$= V\gamma \,.\, \overline{BC} \sin \phi = V\gamma e \sin \phi,$$

and *the centre of buoyancy B does not change its position in the vessel*, since the water-displacing shape remains the same; i.e., no new portions of the vessel are either immersed or raised out of the water.

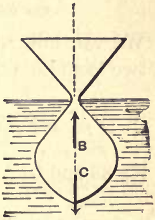

FIG. 510.

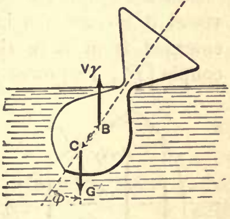

FIG. 511.

But in a vessel of ordinary form, when turned an angle ϕ from the vertical, Fig. 512 (in which ED is a line which is vertical when the ship is upright), there is a *new* centre of buoyancy, B_1, corresponding to the new shape A_1N_1D of the displacement-volume, and the couple to right the vessel (or the reverse) consists of the two forces G' at C and $V\gamma$ at B_1, and has a moment (which we may call M, or *moment of stability*) of a value (§ 28)

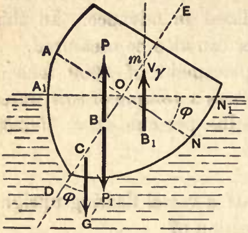

FIG. 512.

$$M = V\gamma \,.\, \overline{mC} \sin \phi. \quad . \quad . \quad (1)$$

Now conceive put in at B (centre of buoyancy of the upright position) two vertical and opposite forces, each $= V\gamma = G'$, calling them P and P_1 (see § 20), Fig. 512. We can now regard the couple $[G', V\gamma]$ as replaced by the two couples $[G', P]$ and $[P_1, V\gamma]$; for evidently

$$V\gamma \,.\, \overline{mC} \sin \phi = V\gamma \,.\, \overline{BC} \sin \phi + V\gamma \,.\, \overline{mB} \sin \phi;$$

(§§ 33 and 34;)

$$\therefore M = V\gamma\, \overline{BC} \sin \phi + V\gamma\, \overline{mB} \sin \phi. \quad . \quad . \quad (2)$$

But the couple $[G', P]$ would be the only one to right the vessel if no new portions of the hull entered the water or emerged from it, in the inclined position; hence the other couple $[P_1, V\gamma]$ owes its existence to the emersion of the wedge AOA_1, and the immersion of the wedge NON_1; i.e., to the loss of a buoyant force $Q =$ (volume $AOA_1) \times \gamma$ on one side, and the gain of an equal buoyant force on the other; therefore this couple $[P_1, V\gamma]$ is the equivalent of the couple $[Q, Q]$, Fig. 513, formed by putting in at the centre of buoyancy of each of the two wedges a vertical force

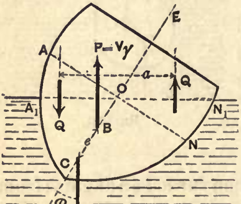

FIG. 513.

$$Q = (\text{vol. of wedge}) \times \gamma = V_w\gamma. \quad (\text{See figure.})$$

If a denotes the arm of this couple, we may write

$$V\gamma\,.\,\overline{mB}\,\sin\phi,\ [\text{of eq. (2)}],\ = V_w\gamma a\,; \quad . \quad . \quad (3)$$

and hence, denoting $\overline{BC}$ by e, we have

$$M = \pm\ V\gamma e\sin\phi + V_w\gamma a\,; \quad . \quad . \quad . \quad . \quad (4)$$

the negative sign in which is to be used when C is above B (as with most ships). O, the intersection of ED and AN, does not necessarily lie on the new water-line plane A_1N_1.

Example.—If a ship of ($V\gamma =$) 3000 tons displacement with C 4 ft. *above* B (i.e., $e = -4$ ft.) is deviated 10° from the vertical, in *salt* water, for which angle the wedges AOA_1 and NON_1 have each a volume of 4000 cubic feet, while the horizontal distance a between their centres of buoyancy is 18 feet, the moment of the acting couple will be, from eq. (4) (ft.-ton-sec. system, in which γ of salt water $= 0.032$),

$$M = -3000\times 4\times 0.1736 + 4000\times 0.032\times 18 = 220.8 \text{ ft. tons},$$

which being + indicates a *righting* couple.

464. Remark.—If with a given ship and cargo this moment of stability, M, be computed, by eq. (4), for a number of values of ϕ, and the results plotted as ordinates (to scale) of a curve, ϕ being the abscissa, the curve obtained is indicative of the general stability of the ship. See Fig. 514. For some value of $\phi = OK$ (as well as for $\phi = 0$) the value of M is zero, and for $\phi > OK$, M is negative, indicating an *upsetting couple*. That is, for $\phi = 0$ the equilibrium is stable, but for $\phi = OK$, *unstable*; and $M = 0$ in both positions. From eq. (4) we see why, if C is above B, instability does not necessarily follow.

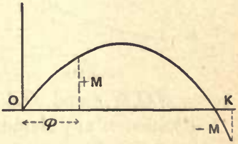

FIG. 514.

464a. Metacentre of a Ship.—Referring again to Fig. 512, we note that the entire couple $[G', V\gamma]$ will be a righting couple, or an upsetting couple, according as the point m (the intersection of the vertical through B_1, the new centre of buoyancy, with BC prolonged) is above or below the centre of gravity C of the ship. The location of this point m changes with ϕ; but as ϕ becomes very small (and ultimately zero) m *approaches a definite position* on the line DE, though not occupying it exactly till $\phi = 0$. This limiting position of m is called the *metacentre*, and accordingly the following may be stated: *A ship floating upright is in stable equilibrium if its metacentre is above its centre of gravity; and vice versâ.* In other words, for a slight inclination from the vertical a righting, and not an upsetting, couple is called into action, if m is above C. To find the metacentre, by means of the distance $\overline{Bm}$, we have, from eq. (3),

$$\overline{mB} = \frac{V_w \gamma a}{V\gamma \sin \phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

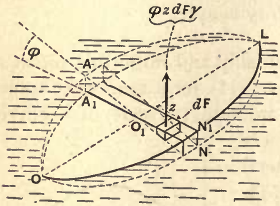

FIG. 515.

and wish ultimately to make $\phi = 0$. Now the moment $(V_w\gamma)a =$ the sum of the moments about the horizontal fore-and-aft water-line axis OL, Fig. 515, of the buoyant efforts due to the immersion of the separate vertical elementary prisms of the wedge OLN_1N, plus the moments of those lost, from emersion, in the wedge OLA_1A. Let OA_1LN_1 be the new water-line section of the ship when inclined a small angle ϕ from the vertical ($\phi = NO_1N_1$), and $OALN$ the old water-line. Let $z =$ the distance of any elementary area dF of the water-line section from OL (which is the intersection of the two water-line planes). Each dF is the base of an elementary prism, with altitude $= \phi z$, of the wedge N_1OLN (or of wedge A_1OLA when z is negative). The buoyant effort of this prism $=$ (its vol.) $\times \gamma = \gamma z\phi dF$, and its moment about OL is $\phi\gamma z^2 dF$. Hence the total moment, $= Qa$, or $V_w\gamma a$, of Fig. 513,

$$= \phi\gamma \int z^2 dF = \gamma\phi \times I_{OL}$$

of water-line section, in which I_{OL} denotes the "moment of inertia" (§ 85) of the plane figure $OALNO$ about the axis OL. Hence from (5), putting $\phi = \sin\phi$ (true when $\phi = 0$), we have $\overline{mB} = I_{OL} \div V$; and therefore the distance $\overline{mC}$, of the metacentre m above C, the centre of gravity of the ship, Fig. 512, is

$$\overline{mC}, = h_m, = \frac{I_{OL}\text{ (of water-line sec.)}}{V} \pm e, \quad . \quad . \quad . \quad (6)$$

in which $e = BC =$ distance from the centre of gravity to the centre of buoyancy, the negative sign being used when C is above B; while $V =$ whole volume of water displaced by the ship.

We may also write, from eqs. (6) and (1), for *small values of* ϕ,

$$\textit{Mom. of righting couple} = M = V\gamma \sin\phi \left[\frac{I_{OL}}{V} \pm e\right], \quad . \quad (7)$$

or

$$M = \gamma \sin\phi\,[I_{OL} \pm Ve]. \quad . \quad . \quad . \quad . \quad (7)'$$

Eqs. (7) and (7)′ will give close approximations for $\phi < 10°$ or 15° with ships of ordinary forms.

EXAMPLE 1.—A homogeneous right parallelopiped, of heaviness γ', floats upright as in Fig. 516. Find the distance $mC = h_m$ for its metacentre in this position, and whether the equilibrium is stable. Here the centre of gravity, C, being the centre of figure, is of course above B, the centre of buoyancy; hence e *is negative.* B is the centre of gravity of the displacement, and is therefore a distance $\frac{1}{2}h$ below the water-line. We here assume that l is greater than b'. From eq. (2), § 461,

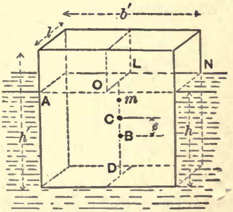

FIG. 516.

$$h = \frac{h'\gamma'}{\gamma};$$

and since $CD = \frac{1}{2}h'$, and $BD = \frac{1}{2}h$, $\therefore\ e = \frac{1}{2}(h' - h)$;

$$\text{i.e., } e = \tfrac{1}{2}h'\left[1 - \frac{\gamma'}{\gamma}\right];$$

while (§ 90) I_{OL}, of the water-line section AN, $= \frac{1}{12}l'b'^3$.

Also,

$$V = b'hl' = b'l'h'\frac{\gamma'}{\gamma};$$

and hence, from eq. (6), we have

$$h_m = \frac{l'b'^3\gamma}{12b'h'l'\gamma'} - \tfrac{1}{2}h'\left[1 - \frac{\gamma'}{\gamma}\right] = \frac{\gamma}{12h'\gamma'}\left[b'^2 - 6h'^2\frac{\gamma'}{\gamma}\left(1 - \frac{\gamma'}{\gamma}\right)\right]$$

Hence if b'^2 is $> 6h'^2\frac{\gamma'}{\gamma}\left(1 - \frac{\gamma'}{\gamma}\right)$, the position in Fig. 516 is one of stable equilibrium, and *vice versâ.* E.g., if $\gamma' = \frac{1}{2}\gamma$, $b' = 12$ inches and $h' = 6$ inches, we have (inch, pound, sec.)

$$h_m = \overline{mC} = \tfrac{1}{36}\left[144 - 6 \times \tfrac{36}{2}(1 - \tfrac{1}{2})\right] = 2.5 \text{ in.}$$

The equilibrium will be unstable if, with $\gamma' = \frac{1}{2}\gamma$, b' is made less than 1.225 h'; for, putting $\overline{mC} = 0$, we obtain $b' = 1.225\ h'$.

Example 2.—(Ft., lb., sec.) Let Fig. 517 represent the *half* water-line section of a loaded ship of $G' = V\gamma = 1010$ tons displacement; required the height of the metacentre above the centre of buoyancy, i.e., $\overline{mB} = ?$ (See equation just before eq. (6).) Now the quantity I_{OL}, of the water-line section, may, from symmetry, (see § 93,) be written

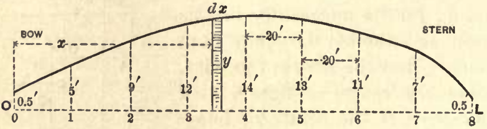

FIG. 517.

$$I_{OL} = 2\int_0^L \tfrac{1}{3}y^3dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $y =$ the ordinate ⏋ to the axis OL at any point; and this, again, by Simpson's Rule for approximate integration, OL being divided into an even number, n, of equal parts, and ordinates erected (see figure), may be written

$$I_{OL} = \frac{2}{3} \cdot \frac{\overline{OL} - 0}{3n} \Big[y_0^3 + 4(y_1^3 + y_3^3 + \dots + y_{n-1}^3) \\ + 2(y_2^3 + y_4^3 + \dots + y_{n-2}^3) + y_n^3 \Big].$$

From which, by numerical substitution (see figure for dimensions; $n = 8$),

$$I_{OL} = \frac{2}{3} \cdot \frac{160}{3 \times 8} \Big[(0.5^3 + 4(5^3 + 12^3 + 13^3 + 7^3) \\ + 2(9^3 + 14^3 + 11^3) + 0.5^3 \Big];$$

or,

$$I_{OL} = \tfrac{40}{9}[0.125 + 4 \times \overset{\begin{array}{r}125\\1728\\2197\\343\end{array}}{\overline{4393}} + 2 \times \overset{\begin{array}{r}729\\2744\\1331\end{array}}{\overline{4804}} + 0.125]$$

$$= 120801 \text{ biquad. ft.}; \therefore \overline{mB} = \frac{I_{OL}}{V} = \frac{120801}{[2020000 \div 64]}$$
$$= 3.8 \text{ feet.}$$

That is, the metacentre is 3.8 feet above the centre of buoyancy, and hence, if $BC = 2$ feet, is 1.90 ft. above the centre of gravity. [See Johnson's Cyclopædia, article *Naval Architecture.*]

465. Metacentre for Longitudinal Stability.—If we consider the stability of a vessel with respect to pitching, in a manner similar to that just pursued for rolling, we derive the position of the metacentre for *pitching* or for longitudinal stability—and this of course occupies a much higher position than that for *rolling*, involving as it does the moment of inertia of the water-line section about a horizontal gravity axis ⏋ to the keel. With this one change, eq. (6) holds for this case also. In large ships the height of this metacentre above the centre of gravity of the ship may be as great as 90 feet.

CHAPTER V.

HYDROSTATICS (*Continued*)—GASEOUS FLUIDS.

466. Thermometers.—The temperature, or "*hotness*," of liquids has, within certain limits, but little influence on their statical behavior, but with gases must always be taken into account, since the three quantities, *tension*, *temperature*, and *volume*, of a given mass of gas are connected by a nearly invariable law, as will be seen.

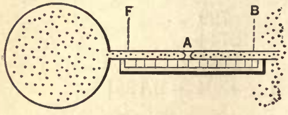

FIG. 518.

An *air-thermometer*, Fig. 518, consists of a large glass bulb filled with air, from which projects a fine straight tube of even bore (so that equal lengths represent equal volumes). A small drop of liquid, *A*, separates the internal from the external air, both of which are at a tension of (say) one atmosphere (14.7 lbs. per sq. inch). When the bulb is placed in melting ice (freezing-point) the drop stands at some point *F* in the tube; when in boiling water (boiling under a pressure of one atmosphere), the drop is found at *B*, on account of the expansion of the internal air under the influence of the heat imparted to it. (The glass also expands, but only about $\frac{1}{150}$ as much; this will be neglected.) The distance *FB* along the tube may now be divided into a convenient number of equal parts called degrees. If into one hundred degrees, it is found that each degree represents a volume equal to the $\frac{367}{100000}$ (.00367) part of the total volume occupied by the air at freezing-point; i.e., the increase of volume from the temperature of freezing-point to that of the boiling-point of water $= 0.367$ of the volume at freezing, *the pressure being the same*, and even having *any value whatever* (as well as one atmosphere), within ordinary limits, so long as it is the same both at freezing and boiling. It must be understood, however, that by *temperature of boiling* is always meant that of water boiling under *one* atmosphere pressure. Another way of stating the above, if one hundred degrees are used between freezing and boiling, is as follows: That for each degree increase of temperature the increase of volume is $\frac{1}{273}$ of the total volume at freezing; 273 being the reciprocal of .00367.

As it is not always practicable to preserve the pressure constant under all circumstances with an air-thermometer, we use the common mercurial thermometer for most practical purposes. In this, the tube is sealed at the outer extremity, with a vacuum above the column of mercury, and its indications agree very closely with those of the air-thermometer. That equal absolute increments of volume should imply equal increments of heat imparted to these thermometric fluids (under constant pressure) could not reasonably be asserted without satisfactory experimental evidence. This, however, is not altogether wanting, so that we are enabled to say that within a moderate range of temperature equal increments of heat produce equal increments of volume in a given mass not only of atmospheric air, but of the so-called "perfect" or "permanent" gases, oxygen, nitrogen, hydrogen, etc. (so named before it was found that they could be liquefied). This is nearly true for mercury also, and for alcohol, *but not for water*. Alcohol has never been frozen, and hence is used instead of mercury as a thermometric substance to measure temperatures below the freezing-point of the latter.

The scale of a mercurial thermometer is fixed; but with an air-thermometer we should have to use a new scale, and in a new position on the tube, for each value of the pressure.

467. Thermometric Scales.—In the *Fahrenheit* scale the tube between freezing and boiling is marked off into 180 equal parts, and the zero placed at 32 of these parts below the freezing-point, which is hence + 32°, and the boiling-point + 212°.

The *Centigrade*, or *Celsius*, scale, which is the one chiefly used in scientific practice, places its zero at freezing, and 100° at boiling-point. Hence to reduce

Fahr. readings to Centigrade, subtract 32° and multiply by $\frac{5}{9}$;
Cent. " " Fahrenheit, multiply by $\frac{9}{5}$ and add 32°.

468. Absolute Temperature.—Experiment also shows that if a mass of air or other perfect gas is confined in a vessel whose volume is but slightly affected by changes of temperature, equal increments of temperature (and therefore equal increments of heat imparted to the gas, according to the preceding paragraph) produce equal increments of tension (i.e., pressure per unit area); or, as to the amount of the increase, that when the temperature is raised by an amount 1° Centigrade, the tension is increased $\frac{1}{273}$ of its value at freezing-point. Hence, theoretically, an ideal barometer (containing a liquid unaffected by changes of temperature) communicating with the confined gas (whose *volume* practically remains constant) would by its indications serve as a thermometer, Fig. 519, and the attached scale could be graduated accordingly. Thus, if the column stood at A when the temperature was freezing, A would be marked 0° on the Centigrade system, and the degree spaces above and below A would each $= \frac{1}{273}$ of the height AB, and therefore the point B (cistern level) to which the column would sink if the gas-tension were zero would be marked $-$ 273° Centigrade.

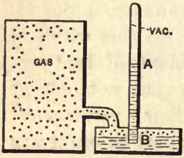

FIG. 519.

But a zero-pressure, in the *Kinetic Theory of Gases* (§ 408), signifies that the gaseous molecules, no longer impinging against the vessel walls (so that the press. = 0), have become motionless; and this, in the Mechanical Theory of Heat, or *Thermodynamics*, implies that *the gas is totally destitute of heat.* Hence this ideal temperature of $-$ 273° Centigrade, or $-$ 460° Fahrenheit, is called the *Absolute Zero of Temperature*, and by reckoning temperatures from it as a starting-point, our formulæ will be rendered much more simple and compact. Temperature so reckoned is called *absolute temperature*, and will be denoted by the letter T. Hence the following rules for reduction:

Absol. temp. T in Cent. degrees = Ordinary Cent. + 273°;
Absol. temp. T in Fahr. degrees = Ordinary Fahr. + 460°.

For example, for 20° Cent., $T = 293°$ Abs. Cent.

469. Distinction Between Gases and Vapors.—All known gases can be converted into liquids by a sufficient reduction of temperature or increase of pressure, or both; some, however, with great difficulty, such as atmospheric air, oxygen, hydrogen, nitrogen, etc., these having been but recently (1878) reduced to the liquid form. A *vapor* is a gas near the point of liquefaction, and does not show that regularity of behavior under changes of temperature and pressure characteristic of a gas when at a temperature much above the point of liquefaction. All gases treated in this chapter (except steam) are supposed in a condition far removed from this stage. The following will illustrate the properties of vapors. See Fig. 520. Let a quantity of liquid, say water, be introduced into a closed space, previously vacuous, of considerably larger volume than the water, and furnished with a manometer and thermometer. Vapor of water immediately begins to form in the space above the liquid, and continues to do so until its pressure attains a definite value dependent on the temperature, and not on the ratio of the volume of the vessel and the original volume of water; e.g., if the temperature is 70° Fahrenheit, the vapor ceases to form when the tension reaches a value of 0.36 lbs. per sq. inch. If heat be gradually applied to raise the temperature, more vapor will form (with ebullition; i.e., from the body of the liquid, unless the heat is applied very slowly), but the tension *will not rise above a fixed value for each temperature* (independent of size of vessel) *so long as there is any liquid left.* Some of these corresponding values, for water, are as follows: For a

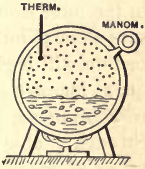

Fig. 520.

Fahr. temp.	= 70°	100°	150°	212°	220°	287°	300°
Tension (lbs. per sq. in.)	= 0.36	0.93	3.69	14.7 = one atm.	17.2	55.0	67.2

At any such stage the vapor is said to be *saturated.*

Finally, at some temperature, dependent on the ratio of the original volume of water to that of the vessel, all of the water will have been converted into vapor (i.e., steam); and if the temperature be still further increased, the tension also increases and *no longer depends on the temperature alone, but also on the heaviness of the vapor when the water disappeared.* The vapor is now said to be *superheated*, and conforms more in its properties to perfect gases.

470. Critical Temperature.—From certain experiments there seems to be reason to believe that at a certain temperature, called the *critical temperature*, different for different liquids, all of the liquid in the vessel (if any remains, and supposing the vessel strong enough to resist the pressure) is converted into vapor, whatever be the size of the vessel. That is, above the critical temperature the substance is necessarily gaseous, in the most exclusive sense, incapable of liquefaction by pressure alone; while below this temperature it is a vapor, and liquefaction will begin if, by compression in a cylinder and consequent increase of pressure, the tension can be raised to a value *corresponding, for a state of saturation, to the temperature* (in such a table as that just given for water). For example, if vapor of water at 220° Fahrenheit and tension of 10 lbs. per sq. inch (this is superheated steam, since 220° is higher than the temperature which for saturation corresponds to $p = 10$ lbs. per sq. inch) is compressed slowly (slowly, to avoid change of temperature) till the tension rises to 17.2 lbs. per sq. in., which (see above table) is the pressure of saturation for a temperature of 220° Fahrenheit for water-vapor, the vapor is saturated, i.e. liquefaction is ready to begin, and during any further slow reduction of volume the pressure remains constant and some of the vapor is liquefied.

By "perfect gases," or gases proper, we may understand, therefore, those which cannot be liquefied by pressure unaccompanied by great reduction of temperature; i.e., whose "critical temperatures" are very low. The critical temperature of N_2O, or nitrous oxide gas, is between $-11°$ and $+8°$ Centigrade, while that of oxygen is said to be at $-118°$ Centigrade. [See p. 471, vol. 122 of the *Journal of the Franklin Institute.* For an account of the liquefaction of oxygen, etc., see the same periodical, January to June, 1878.]

471. Law of Charles (and of Gay Lussac).—The mode of graduation of the air-thermometer may be expressed in the following formula, which holds good (for practical purposes) within the ordinary limits of experiment for a given mass of *any perfect gas, the tension remaining constant:*

$$V = V_0 + 0.00367\,V_0 t = V_0(1 + .00367t); \quad . \quad . \quad (1)$$

in which V_0 denotes the volume occupied by the given mass at freezing-point under the given pressure, V its volume at any other temperature t Centigrade under the *same pressure.* Now, 273 being the reciprocal of .00367, we may write

$$V = V_0\frac{(273 + t)}{273}; \text{ i.e., } \frac{V}{V_0} = \frac{T}{T_0} \quad . \quad . \quad \left\{\begin{matrix}\text{press.}\\ \text{const.}\end{matrix}\right\}; \quad (2)$$

(see § 468;) in which $T_0 =$ the *absolute temperature* of freezing-point, $= 273°$ absolute Centigrade, and T the absolute temperature corresponding to t Centigrade. Eq. (2) is also true when T and T_0 are both expressed in Fahrenheit degrees (from absolute zero, of course). Accordingly, we may say that, *the pressure remaining the same, the volume of a given mass of gas varies directly as the absolute temperature.*

Since the weight of the given mass of gas is invariable at a given place on the earth's surface, we may

always use the equation $V\gamma = V_0\gamma_0$, ... (3)

pressure constant or not, and hence (2) may be rewritten

$$\frac{\gamma_0}{\gamma} = \frac{T}{T_0} . . . \text{(press. const.)}; \quad . \quad (4)$$

i.e., *if the pressure is constant, the heaviness (and therefore the specific gravity) varies inversely as the absolute temperature.*

Experiment also shows (§ 468) that if the volume [and therefore the heaviness, eq. (3)] remains constant, while the temperature varies, the tension p will change according to the following relation, in which p_0 = the tension when the temperature is freezing:

$$p = p_0 + \tfrac{1}{273} p_0 t = p_0 \frac{273 + t}{273}, \quad . \quad . \quad . \quad (5)$$

t denoting the Centigrade temperature. Hence transforming, as before, we have

$$\frac{p}{p_0} = \frac{T}{T_0} \quad . \quad . \quad \left\{ \begin{matrix} \text{vol., and } \therefore \\ \text{heav., const.} \end{matrix} \right\}; \quad . \quad (6)$$

or, *the volume and heaviness remaining constant, the tension of a given mass of gas varies directly as the absolute temperature.* This is called the *Law of Charles* (or of *Gay Lussac*).

472. General Formulæ for any Change of State of a Perfect Gas.—If any two of the three quantities, viz., *volume* (or heaviness), *tension*, and *temperature*, are changed, the new value of the third is determinate from those of the other two, according to a relation proved as follows (remembering that *henceforth the absolute temperature only* will be used, T, § 468): Fig. 521. At A a certain mass of gas at a tension of p_0, one atmosphere, and absolute temperature T_0 (freezing), occupies a volume V_0. Let it now be heated to an absolute temp. $= T'$, without change of tension (expanding behind a piston, for instance). Its volume will increase to a value V which from (2) of § 471 will satisfy the relation

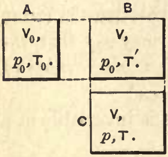

FIG. 521.

$$\frac{V}{V_0} = \frac{T'}{T_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(See B in figure.)

Let it now be heated without change of volume to an absolute temperature T (C in figure). Its volume is still V, but the tension has risen to a value p, such that, on comparing B and C by eq. (6), we have

$$\frac{p}{p_0} = \frac{T}{T'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Combining (7) and (8), we obtain for *any state* in which the tension is p, volume V, and absolute temperature T, in

$$(General) \quad . \quad . \quad . \quad \frac{pV}{T} = \frac{p_0 V_0}{T_0}; \quad \text{or } \frac{pV}{T} = a \; constant; \quad . \quad (9)$$

or

$$(General) \quad . \quad . \quad . \quad . \quad \frac{p_m V_m}{T_m} = \frac{p_n V_n}{T_n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which, since

$$(General) \quad . \quad . \quad V\gamma = V_0\gamma_0 = V_m\gamma_m = V_n\gamma_n, \quad . \quad . \quad . \quad (11)$$

is true for any change of state, we may also write

$$(General) \quad . \quad . \quad . \quad . \quad \frac{p}{\gamma T} = \frac{p_0}{\gamma_0 T_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

or

$$\frac{p_m}{\gamma_m T_m} = \frac{p_n}{\gamma_n T_n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These equations (9) to (13), inclusive, hold good for any state of a mass of any perfect gas (most accurately for air). The subscript 0 refers to the state of one-atmosphere tension and freezing-point temperature, m and n to any two states whatever (within practical limits); γ is the heaviness, §§ 7 and 409, and T the *absolute temperature*, § 468.

If p, V, and T of equation (9) be treated as variables, and laid off to scale as co-ordinates parallel to three axes in space, respectively, the surface so formed of which (9) is the equation is a hyperbolic paraboloid.

473. Examples.—EXAMPLE 1.—What cubic space will be occupied by 2 lbs. of hydrogen gas at a tension of two atmospheres and a temperature of 27° Centigrade?

With the *inch-lb.-sec.* system we have $p_0 = 14.7$ lbs. per sq. inch, $\gamma_0 = [.0056 \div 1728]$ lbs. per cubic inch, and $T_0 = 273°$ absolute Centigrade, when the gas is at freezing-point at one atmosphere (i.e., in *state sub-zero*). In the state mentioned in the problem, we have $p = 2 \times 14.7$ lbs. per sq. in.,

$$T = 273 + 27 = 300° \text{ absolute Centigrade,}$$

while γ is required. Hence, from eq. (12),

$$\frac{2 \times 14.7}{\gamma\ 300} = \frac{14.7}{(.0056 \div 1728)273};$$

$\therefore \gamma = \dfrac{.0102}{1728}$ lbs. per cub. in. $= .0102$ lbs. per cub. foot; and if the total weight, $= G, = V\gamma$, is to be 2 lbs., we have (ft., lb., sec.) $V = 2 \div 0102 = 196$ cubic feet.—*Ans.*

EXAMPLE 2.—A mass of air originally at 24° Centigrade and a tension indicated by a barometric column of 40 inches of mercury has been simultaneously reduced to half its former volume and heated to 100° Centigrade; required its tension in this new state, which we call the state n, m being the original state. Use the inch, lb., sec. We have given, therefore, $p_m = \frac{40}{30} \times 14.7$ lbs. per sq. inch, $T_m = 273 + 24 = 297°$ absolute Centigrade, the ratio

$$V_m : V_n = 2 : 1, \text{ and } T_n = 273 + 100 = 373° \text{ Abs. Cent.;}$$

while p_n is the unknown quantity. From eq. (10), hence,

$$p_n = \frac{V_m}{V_n} \cdot \frac{T_n}{T_m} \cdot p_m = 2 \times \tfrac{373}{297} \cdot \tfrac{40}{30} \times 14.7 = 49.22 \text{ lbs. per sq. in.,}$$

which an ordinary steam-gauge would indicate as

$$(49.22 - 14.7) = 34.52 \text{ lbs. per sq. inch.}$$

(That is, if the weather barometer indicated exactly 14.7 lbs per sq. inch.)

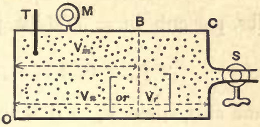

FIG. 522.

EXAMPLE 3.—A mass of air, Fig. 522, occupies a rigid closed vessel at a temperature of 15° Centigrade (equal to that of surrounding objects) and a tension of four atmospheres [*state m*]. By opening a stop-cock a few seconds, thus allowing a portion of the gas to escape quickly, and then shutting it, the remainder of the air [now in *state n*] is found to have a tension of only 2.5 atmospheres (measured immediately); its temperature cannot be measured immediately (so much time being necessary to affect a thermometer), and is less than before. To compute this temperature, T_n, we allow the air now in the vessel to come again to the same temperature as surrounding objects (15° Centigrade); find then the tension to be 2.92 atmospheres. Call the last state, *state r* (inch, lb., sec.). The problem then stands thus:

$p_m = 4 \times 14.7$	$p_n = 2.5 \times 14.7$	$p_r = 2.92 \times 14.7$
$\gamma_m = ?$	$\gamma_n = ?$	$\gamma_r = \gamma_n$ (since $V_r = V_n$)
$T_m = 288°$ Abs. Cent.	$T_n = ?\left\{\begin{array}{l}\text{principal}\\ \text{unknown}\end{array}\right.$	$T_r = T_m = 288°$ Abs. Cent.

In states n and r the heaviness is the same; hence an equation like (6) of § 471 is applicable, whence

$$\frac{p_n}{p_r} = \frac{T_n}{T_r}, \quad \text{or } T_n = \frac{2.5 \times 14.7}{2.92 \times 14.7} \times 288 = 246° \text{ Abs. Cent.}$$

or — 27° Centigrade; considerably *below freezing*, as a result of allowing the sudden escape of a portion of the air, and the consequent sudden expansion, and reduction of tension, of the remainder. In this sudden passage from state m to state n, the remainder altered its heaviness (and its volume in inverse ratio) in the ratio (see eqs. (11) and (10) of § 472)

$$\frac{\gamma_n}{\gamma_m} = \frac{V_m}{V_n} = \frac{p_n}{p_m} \cdot \frac{T_m}{T_n} = \frac{2.5 \times 14.7}{4 \times 14.7} \cdot \frac{288}{246} = 0.73.$$

Now the heaviness in state m (see eq. (12), § 472) was

$$\gamma_m = \frac{p_m}{T_m} \cdot \frac{\gamma_0 T_0}{p_0} = \frac{4 \times 14.7}{288} \cdot \frac{.0807}{1728} \cdot \frac{273}{14.7} = \frac{.306}{1728}$$

lbs. per cub. in. = .306 lbs. per cub. ft.

$$\therefore \gamma_n = 0.73 \times \gamma_m = 0.223 \text{ lbs. per cub. ft.,}$$

and also, since $V_m = 0.73\, V_n$, about $\frac{27}{100}$ of the original quantity of air in vessel has escaped.

[NOTE.—By numerous experiments like this, the law of cooling, when a mass of gas is allowed to expand suddenly (as, e.g., behind a piston, doing work) has been determined; and *vice versâ*, the law of heating under sudden compression: see § 487.]

474. The Closed Air-manometer.—If a manometer be formed of a straight tube of glass, of uniform cylindrical bore, which is partially filled with mercury and then inverted in a cistern of mercury, *a quantity of air having been left between the mercury and the upper end of the tube, which is closed*, the tension of this confined air (to be computed from its observed volume and temperature) must be added to that due to the mercury column, in order to obtain the tension p' to be measured. See Fig. 523. The advantage of this kind of instrument is, that to measure great tensions the tube need not be very long. Let the temperature T_1 of whole instrument, and the tension p_1 of the air or gas in the cistern, be known when the mercury in the tube stands at the same level as that in the cistern. The tension of the air in the tube must now be p_1 also, its temperature T_1, and its volume is $V_1 = Fh_1$, F being the sectional area of the bore of the tube; see on left of figure. When the instrument is used, gas of unknown tension p' is admitted to the cistern, the temperature of the whole instrument being noted ($= T$), and the heights h and h'' are observed ($h + h''$ cannot be put $= h_1$

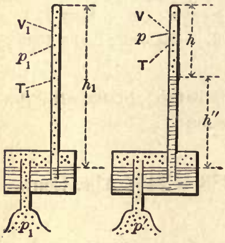

FIG. 523.

unless the cistern is very large). p' is then computed as follows (eq. (2), § 413):

$$p' = h'' \gamma_m + p \,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $p =$ the tension of the air in the tube, and γ_m the heaviness of mercury. But from eq. (10), § 472, putting $V_1 = Fh_1$ and $V = Fh$,

$$p = p_1 \frac{V_1}{V} \cdot \frac{T}{T_1} = \frac{h_1}{h} \frac{T}{T_1} p_1. \quad . \quad . \quad . \quad . \quad (2)$$

Hence finally, from (1) and (2),

$$p' = h'' \gamma_m + \frac{h_1}{h} \cdot \frac{T}{T_1} p_1. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since T_1, p_1, and h_1 are fixed constants for each instrument, we may, from (3), compute p' for any observed values of h and T (N.B. T and T_1 are *absolute temperatures*), and construct a series of tables each of which shall give values of p' for a range of values of h, and one special value of T.

EXAMPLE.—Supposing the fixed constants of a closed air-manometer to be (in inch-lb.-sec. system) $p_1 = 14.7$ (or one atmosphere), $T_1 = 285°$ Abs. Cent. (i.e., 12° Centigrade), and $h_1 = 3' \, 4'' = 40$ inches; required the tension in the cistern indicated by $h'' = 25$ inches and $h = 15$ inches, when the temperature is $-3°$ Centigrade, or $T = 270°$ Abs. Cent.

For mercury, $\gamma_m = [848.7 \div 1728]$ (§ 409) (though strictly it should be specially computed for the temperature, since it varies about .00002 of itself for each Centigrade degree). Hence, eq. (3),

$$p' = \frac{25 \times 848.7}{1728} + \frac{40}{15} \cdot \frac{270}{285} \times 14.7 = 12.26 + 37.13 = 49.39$$

lbs. per sq. inch, or nearly $3\frac{1}{3}$ atmospheres [steam-gauge would read 34.7 lbs. per sq. in.].

475. Mariotte's Law, (or Boyle's,) Temperature Constant; i.e., Isothermal Change.—If a mass of gas be compressed, or allowed to expand, *isothermally*, i.e., without change of temperature (practically this cannot be done unless the walls of the vessel are conductors of heat, and then the motion must be slow), eq. (10) of § 472 now becomes (since $T_m = T_n$)

$$\left\{\begin{array}{c}\textit{Mariotte's Law,}\\ \textit{Temp. constant}\end{array}\right\} \quad . \quad V_m p_m = V_n p_n, \text{ or } \frac{p_m}{p_n} = \frac{V_n}{V_m}; \quad . \quad (1)$$

i.e., *the temperature remaining unchanged, the tensions are inversely proportional to the volumes, of a given mass of a perfect gas;* or, *the product of volume by tension is a constant quantity.* Again, since $V_m \gamma_m = V_n \gamma_n$ for any change of state,

$$\left\{\begin{array}{c}\textit{Mariotte's Law,}\\ \textit{Temp. constant}\end{array}\right\} \quad . \quad . \quad \frac{p_m}{p_n} = \frac{\gamma_m}{\gamma_n}, \text{ or } \frac{p_m}{\gamma_m} = \frac{p_n}{\gamma_n}; \quad . \quad (2)$$

i.e., *the pressures (or tensions are directly proportional to the (first power of the) heavinesses, if the temperature is the same.*

This law, which is very closely followed by all the perfect gases, was discovered by Boyle in England and Mariotte in France more than two hundred years ago, but of course is only a particular case of the general formula, for any change of state, in § 472. It may be verified experimentally in several ways. E.g., in Fig. 524, the tube OM being closed at the top, while PN is open, let mercury be poured in at P until it reaches the level $A'B'$. The air in OA' is now at a tension of one atmosphere. Let more mercury be slowly poured in at P, until the air confined in O has been compressed to a volume $OA'' = \frac{1}{2}$ of OA', and the height $B''E''$ then measured; it will be found to be 30 inches; i.e., the tension of the air in O is now *two atmospheres* (corresponding to 60 inches of mercury). Again, compress the air in O to $\frac{1}{3}$ its original volume (when at one atmosphere), i.e., to volume $OA''' = \frac{1}{3}OA'$, and the mercury height $B'''E'''$ will be 60 inches, showing a tension of three atmospheres in the confined air at O (90

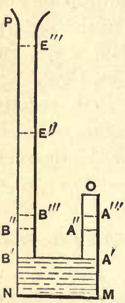

FIG. 524.

inches of mercury in a barometer). It is understood that the temperature is the same, i.e., that time is given the compressed air to acquire the temperature of surrounding objects after being heated by the compression, if sudden.

[NOTE.—The law of decrease of steam-pressure in a steam-engine cylinder, after the piston has passed the point of "cut-off" and the confined steam is expanding, does not materially differ from Mariotte's law, which is often applied to the case of expanding steam; see § 479.]

While Mariotte's law may be considered exact for practical purposes, it is only approximately true, the amount of the deviations being different at different temperatures. Thus, for decreasing temperatures the product Vp of volume by tension becomes smaller, with most gases.

EXAMPLE 1.—If a mass of compressed air expands in a cylinder behind a piston, having a tension of 60 lbs. per sq. inch (45.3 by steam-gauge) at the beginning of the expansion, which is supposed slow (that the temperature may not fall); then when it has doubled in volume its tension will be only 30 lbs. per sq. inch; when it has tripled in volume its tension will be only 20 lbs. per sq. inch, and so on.

EXAMPLE 2. *Diving-bell.*—Fig. 525. If the cylindrical diving-bell AB is 10 ft. in height, in what depth, $h = ?$, of *salt water*, can it be let down to the bottom, without allowing the water to rise in the bell more than a distance $a = 4$ ft.? Call the horizontal sectional area, F. The mass of air in the bell is constant, at a constant temperature. *First, algebraically;* at the surface this mass of air occupied a volume $V_m = Fh''$ at a tension $p_m = 14.7 \times 144$ lbs. per sq. ft., while at the depth mentioned it is compressed to a volume $V_n = F(h'' - a)$, and is at a tension $p_n = p_m + (h - a)\gamma_w$, in which $\gamma_w =$ heaviness of salt water. Hence, from eq. (1),

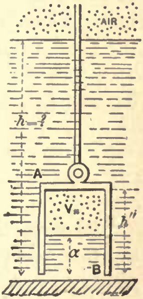

FIG. 525.

$$p_m Fh'' = [p_m + (h - a)\gamma_w]F(h'' - a); \quad . \quad . \quad (3)$$

$$\therefore\ h = a\left[1 + \frac{p_m}{(h'' - a)\gamma_w}\right]; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

hence, *numerically*, (ft., lb., sec.,)

$$h = 4 \times \left[1 + \frac{14.7 \times 144}{(10 - 4) \times 64}\right] = 26.05 \text{ feet.}$$

476. Mixture of Gases.—It is sometimes stated that if a vessel is occupied by a mixture of gases (between which there is no chemical action), the tension of the mixture is equal to the sum of the pressures of each of the component gases present; or, more definitely, is equal to the sum of the pressures which the separate masses of gas would exert on the vessel if each in turn occupied it alone at the same temperature.

This is a direct consequence of Mariotte's law, and may be demonstrated as follows:

Let the actual tension be p, and the capacity of the vessel V. Also let V_1, V_2, etc., be the volumes actually occupied by the separate masses of gas, so that

$$V_1 + V_2 + \ldots = V; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and p_1, p_2, etc., the pressures they would individually exert when occupying the volume V alone at the same temperature. Then, by Mariotte's law,

$$Vp_1 = V_1p; \quad Vp_2 = V_2p; \quad \text{etc.}; \quad . \quad . \quad . \quad (2)$$

whence, by addition, we have

$$V(p_1 + p_2 + \ldots) = (V_1 + V_2 + \ldots)p;$$

$$\text{i.e.,} \quad p = p_1 + p_2 + \ldots \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Of course, the same statement applies to any number of separate parts into which we may imagine a mass of homogeneous gas to be divided.

For numerical examples and practical questions in the solution of which this principle is useful, see p. 239, etc., Rankine's Steam-engine. (Rankine uses 0.365, where 0.367 has been used here.)

477. Barometric Levelling.—By measuring with a barometer the tension of the atmosphere at two different levels, simultaneously, and on a still day, the two localities not being widely separated horizontally, we may compute their vertical distance apart if the temperature of the stratum of air between them is known, being the same, or nearly so, at both stations. Since the heaviness of the air is different in different layers of the vertical column between the two elevations N and M, Fig. 526, we cannot immediately regard the whole of such a column as a free body (as was done with a liquid, § 412), but must consider a horizontal thin lamina, L, of thickness $\doteq dz$ and at a distance $= z$ (variable) below M, the level of the upper station, N being the lower level at a distance, h, from M.

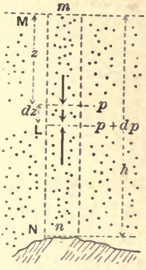

FIG. 526.

The tension, p, must increase from M downwards, since the lower laminæ have to support a greater weight than the upper; and the heaviness γ must also increase, proportionally to p, since we assume that all parts of the column are at the same temperature, thus being able to apply Mariotte's law. Let the tension and heaviness of the air at the upper base of the elementary lamina, L, be p and γ respectively. At the lower base, a distance dz below the upper, the tension is $p + dp$. Let the area of the base of lamina be F; then the vertical forces acting on the lamina are Fp, downward; its weight $\gamma F dz$ downward; and $F(p + dp)$ upward. For its equilibrium Σ(vert. compons.) must $= 0$;

$$\therefore F(p + dp) - Fp - F\gamma dz = 0;$$

$$\text{i.e., } dp = \gamma dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which contains three variables. But from Mariotte's law, § 475, eq. (2), if p_n and γ_n refer to the air at N, we may substitute $\gamma = \frac{\gamma_n}{p_n} p$ and obtain, after dividing by p, to separate the variables p and z,

$$\frac{p_n}{\gamma_n} \cdot \frac{dp}{p} = dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Summing equations like (2), one for each lamina between M (where $p = p_m$ and $z = 0$) and N (where $p = p_n$ and $z = h$), we have

$$\frac{p_n}{\gamma_n} \cdot \int_{p_m}^{p_n} \frac{dp}{p} = \int_0^h dz;$$

$$\text{i.e., } h = \frac{p_n}{\gamma_n} \ \log._e \left[\frac{p_n}{p_m}\right], \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives h, the difference of level, or altitude, between M and N, in terms of the observed tensions p_n and p_m, and of γ_n, the heaviness of the air at N, which may be computed from eq. (12), § 472, substituting from which we have finally

$$h = \frac{p_0}{\gamma_0} \cdot \frac{T_n}{T_0} \cdot \log._e \left[\frac{p_n}{p_m}\right], \quad . \quad . \quad . \quad . \quad (4)$$

in which the subscript 0 refers to freezing-point and one atmosphere tension; T_n and T_0 are absolute temperatures. For the ratio $p_n : p_m$ we may put the equal ratio $h_n : h_m$ of the actual barometric heights which measure the tensions. The log. $_e$(or Naperian, or natural, or hyperbolic, log.) = (common log. to base 10) × 2.30258. From § 394, γ_0 of air = 0.08076 lbs. per cub. ft., and $p_0 = 14.701$ lbs. per sq. inch; $T_0 = 273°$ Abs. Cent.

If the temperature of the two stations (both in the *shade*) are not equal, a mean temp. $= \frac{1}{2}(T_m + T_n)$ may be used for T_n in eq. (4), for approximate results. Eq. (4) may then be written

$$h \text{ (in feet)} = 26213 \frac{T_n}{T_0} \cdot \log._e \left[\frac{p_n}{p_m}\right]. \quad . \quad . \quad . \quad (5)$$

The quantity $\frac{p_0}{\gamma_0} = 26213$ ft., just substituted, is called the *height of the homogeneous atmosphere*, i.e., the ideal height which the atmosphere would have, if incompressible and nonexpansive like a liquid, in order to exert a pressure of 14.701 lbs. per sq. inch upon its base, being throughout of a constant heaviness = .08076 lbs. per cub. foot.

By inversion of eq. (4) we may also write

$$p_m e^{\frac{\gamma_0}{p_0} \cdot \frac{T_0}{T_n} \cdot h} = p_n, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $e = 2.71828$ = the Naperian Base, which is to be raised to the power whose index is the abstract number $\frac{\gamma_0}{p_0} \cdot \frac{T_0}{T_n} \cdot h$, and the result multiplied by p_m to obtain p_n.

Example.—Having observed as follows (simultaneously):

At lower station N, $h_n = 30.05$ in. mercury; temp. = 77.6° F.;
" upper " M, $h_m = 23.66$ " " " = 70.4° F.;

required the altitude h. From these figures we have a mean absolute temperature of $460° + \frac{1}{2}(77.6 + 70.4) = 534°$ Abs. Fahr.; hence, from (5),

$$h = 26213 \times \tfrac{534}{492} \times 2.30258 \times \log._{10} \left[\frac{30.05}{23.66}\right] = 6787.9 \text{ ft.}$$

(Mt. Guanaxuato, in Mexico, by Baron von Humboldt.) Strictly, we should take into account the latitude of the place, since γ_0 varies with g (see § 76), and also the decrease in the intensity of gravitation as we proceed farther from the earth's centre, for the mercury in the barometer weighs less per cubic inch at the upper station than at the lower.

Tables for use in barometric levelling can be found in Trautwine's Pocket-book, and in Searles's Field-book for Railroad Engineers, as also tables of boiling-points of water under different atmospheric pressures, forming the basis of another method of determining heights.

478. Adiabatic Change—Poisson's Law.—By an *adiabatic* change of state, on the part of a gas, is meant a compression or expansion in which work is done *upon* the gas (in compressing it) or *by* the gas (in expanding against a resistance) when there is *no transmission of heat* between the gas and enclosing vessel, or surrounding objects, by conduction or radiation. This occurs when the volume changes in a vessel of non-conducting material, or when the compression or expansion takes place *so quickly* that there is no time for transmission of heat to or from the gas.

The experimental facts are, that if a mass of gas in a cylinder be suddenly compressed to a smaller volume its temperature is raised, and its tension increased more than the change of volume would call for by Mariotte's law; and *vice versâ*, if a gas at high tension is allowed to expand in a cylinder and drive a piston against a resistance, its temperature falls, and its tension diminishes more rapidly than by Mariotte's law.

Again (see Example 3, § 473), if $\frac{27}{100}$ of the gas in a rigid vessel, originally at 4 atmos. tension and temperature of 15° Cent., is allowed to escape suddenly through a stop-cock into the outer air, the remainder, while increasing its volume in the ratio 100 : 73, is found to have cooled to $-$ 27° Cent., and its tension to have fallen to 2.5 atmospheres; whereas, by Mariotte's law, if the temperature had been kept at 288° Abs. Cent., the tension would have been lowered to $\frac{73}{100}$ of 4, i.e., to 2.92 atmospheres only.

The reason for this cooling during sudden expansion is, according to the *Kinetic Theory of Gases*, that since the "sensible heat" (i.e., that perceived by the thermometer), or "*hotness*" of a gas depends on the velocity of its incessantly moving molecules, and that each molecule after impact with a *receding* piston has a less velocity than before, the temperature necessarily falls; and *vice versâ*, when an advancing piston compresses the gas into a smaller volume.

If, however, a mass of gas expands *without doing work*, as when, in a vessel of two chambers, one a vacuum, the other full of gas, communication is opened between them, and the gas allowed to fill both chambers, *no cooling* is noted in the mass as a whole (though parts may have been cooled temporarily).

By experiments similar to that in Example 3, § 473, it has been found that for air and the "perfect gases," in an adiabatic change of volume [and therefore of heaviness], the tension varies inversely with the 1.41th power of the volume. This is called *Poisson's Law.* For ordinary purposes (as Weisbach suggests) we may use $\frac{3}{2}$ instead of 1.41, and hence write

$$\left.\begin{matrix}\textit{Adiabat.}\\ \textit{Change}\end{matrix}\right\} \quad . \quad . \quad \frac{p_m}{p_n} = \left(\frac{\gamma_m}{\gamma_n}\right)^{\frac{3}{2}}, \quad . \quad . \quad \text{or } \frac{p_m}{p_n} = \left(\frac{V_n}{V_m}\right)^{\frac{3}{2}}; \quad . \quad . \quad (1)$$

and combining this relation with the general eqs. (10) and (13), § 472, we have also

$$\left.\begin{matrix}\textit{Adiabat.}\\ \textit{Change}\end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad \frac{p_m}{p_n} = \left(\frac{T_m}{T_n}\right)^{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

i.e., the tension varies directly as the cube of the absolute temperature; also,

$$\left.\begin{matrix}\textit{Adiabat.}\\ \textit{Change}\end{matrix}\right\} \quad . \quad \left(\frac{V_m}{V_n}\right) = \left(\frac{T_n}{T_m}\right)^{2}, \quad \text{or } \frac{\gamma_n}{\gamma_m} = \left(\frac{T_n}{T_m}\right)^{2}; \quad . \quad . \quad . \quad (3)$$

i.e., the volume is inversely, and the heaviness directly, as the square of the absolute temperature.

Here m and n refer to any two adiabatically related states. T is the *absolute temperature.*

EXAMPLE 1.—Air in a cylinder at 20° Cent. is *suddenly* compressed to $\frac{1}{6}$ its original volume (and therefore is six times as dense, i.e., has six times the heaviness, as before). To what temperature is it heated? Let m be the initial state, and n the final. From eq. (3) we have

$$\frac{T_n}{293} = \sqrt{\frac{6}{1}}; \quad \therefore\ T_n = 718^\circ \text{ Abs. Cent.,}$$

or nearly double the absolute temperature of boiling water.

EXAMPLE 2.—After the air in Example 1 has been given time to cool again to 20° Cent. (temperature of surrounding objects) it is allowed to resume, suddenly, its first volume, i.e., to increase its volume sixfold by expanding behind a piston. To what temperature has it cooled? Here $T_m = 293°$ Abs. Cent., the ratio $V_m : V_n = \frac{1}{6}$, and T_n is required. Hence, from (3),

$$\frac{T_n}{293} = \sqrt{\frac{1}{6}}; \quad \therefore T_n = 293 \div \sqrt{6} = 119.5° \text{ Abs. Cent.,}$$

or $= -154°$ Cent., indicating extreme cold.

From these two examples the principle of one kind of ice-making apparatus is very evident. As to the work necessary to compress the air in Example 1, see § 483. It is also evident why motors using compressed air expansively have to encounter the difficulty of frozen watery vapor (present in the air to some extent).

Example 3.—What is the tension of the air in Example 1 (suddenly compressed to $\frac{1}{6}$ its original volume) immediately after the compression, if the original tension was one atmosphere? That is, with $V_n : V_m = 1 : 6$, and $p_m = 14.7$ lbs. per sq. inch, $p_n = ?$ From eq. (1), (in., lb., sec.,)

$$p_n = 14.7 \times 6^{\frac{3}{2}} = 14.7 \sqrt{216} = 216$$

lbs. per sq. inch; whereas, if, after compression and without change of volume, it cools again to 20° Cent., the tension is only $14.7 \times 6 = 88.2$ lbs. per sq. inch (now using Mariotte's law).

479. Work of Expanding Steam following Mariotte's Law.—Although gases do not in general follow Mariotte's law in expanding behind a piston (without special provision for supplying heat), it is found that the tension of saturated steam (i.e., saturated at the beginning of the expansion) in a steam-engine cylinder, when left to expand after the piston has passed the point of "*cut-off*," diminishes very nearly in accordance with Mariotte's law, which may therefore be applied in this case to find the work done per stroke, and thence the power. In Fig. 527 a horizontal steam-cylinder is shown in which the piston is making its left-to-right stroke. The "back-pressure" is constant and $= Fq$, F being the area of the piston and q the *intensity* (i.e., per unit area) of the back or exhaust pressure on the right side of the piston; while the forward pressure on the left face of the piston $= Fp$, in which p is the steam-pressure per unit area, and is different at different points of the stroke. While the piston is passing from O'' to D'', p is constant, being $= p_b =$ the boiler-pressure, since the steam-port is still open. Between D'' and C'', however, the steam being *cut off* (i.e., the steam-port is closed) at D'', a distance a from O'', p decreases with Mariotte's law (nearly), and its value is $(Fa \div Fx)p_b$ at any point on $C''D''$, x being the distance of the point from O''.

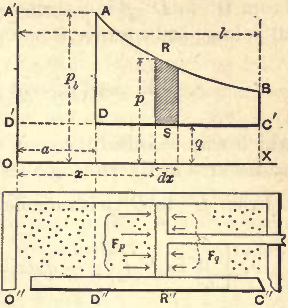

FIG. 527.

Above the cylinder, conceive to be drawn a diagram in which an axis OX is ‖ to the cylinder-axis, OY an axis ⏋ to the same, while O is vertically above the left-hand end of the cylinder. As the piston moves, let the value of p corresponding to each of its positions be laid off, to scale, in the vertical immediately above the piston, as an ordinate from the axis X. Make $OD' = q$ by the same scale, and draw the horizontal $D'C'$. Then the effective work done on the piston-rod while it moves through any small distance dx is

$$dW = \text{force} \times \text{distance} = F(p - q)dx,$$

and is proportional to the area of the strip RS, whose width is dx and length $= p - q$; so that the effective work of one stroke is

$$\Big[_{O''}^{C''} W = \int_{x=0}^{x=l} dW = F\int_{x=0}^{x=l}(p - q)dx, \quad . \quad . \quad . \quad (1)$$

and is represented graphically by the area $A'ARBC'D'A'$ From O'' to D'' p is constant and $=p_b$ (while q is constant at all points), and x varies from 0 to a;

$$\therefore \left[\overset{D''}{\underset{O''}{W}}\right. = F(p_b - q)\int_0^a dx = F(p_b - q)a, \quad . \quad . \quad (2)$$

which may be called the *work of entrance*, and is represented by the area of the rectangle $A'ADD'$.

From D'' to C'' p is variable and, by Mariotte's law, $= \frac{a}{x}p_b$;

$$\therefore \left[\overset{C''}{\underset{D''}{W}}\right. = F\int_a^l (p - q)dx = F\left[ap_b\int_a^l \frac{dx}{x} - q\int_a^l dx\right];$$

i.e.,

$$\left[\overset{C''}{\underset{D''}{W}}\right. = F\left[ap_b \log._e \left(\frac{l}{a}\right) - q(l-a)\right] \quad . \quad . \quad . \quad (3)$$

$=$ the *work of expansion*, adding which to that of entrance, we have for the *total effective work of one stroke*

$$W = Fp_b a\left[1 + \log._e \left(\frac{l}{a}\right)\right] - Fql. \quad . \quad . \quad . \quad (4)$$

By effective work we mean that done upon the piston-rod and thus transmitted to outside machinery. Suppose the engine to be "double-acting"; then at the end of the stroke a communication is made, by motion of the proper valves, between the space on the left of the piston and the condenser of the engine; and also between the right of the piston and the boiler (that to the condenser now being closed). On the return stroke, therefore, the conditions are the same as in the forward stroke, except that the two sides of the piston have changed places as regards the pressures acting on them, and thus the same amount of effective work is done as before.

If n revolutions of the fly-wheel are made per unit of time (two strokes to each revolution), the effective work done per unit of time, i.e., the *power* of the engine, is

$$L = 2nW = 2nF\left[ap_b\left[1 + \log._e \left(\frac{l}{a}\right)\right] - ql\right]. \quad (5)$$

For simplicity the above theory has omitted the consideration of "*clearance*," that is, the fact that at the point of "cut-off" the mass of steam which is to expand occupies not only the cylindrical volume Fa, but also the "clearance" or small space in the steam-passages between the valve and the entrance of the cylinder, the space between piston and valve which is never encroached upon by the piston. "Wire-drawing" has also been disregarded, i.e., the fact that during communication with the boiler the steam-pressure on the piston is a little less than boiler-pressure. For these the student should consult special works, and also for the consideration of water mixed with the steam, etc. Again, a strict analysis should take into account the difference in the areas which receive fluid-pressure on the two sides of the piston.

EXAMPLE 1.—A reciprocating steam-engine makes 120 revolutions per minute, the boiler-pressure is 40 lbs. by the gauge (i.e., $p_b = 40 + 14.7 = 54.7$ lbs. per sq. inch), the piston area is $F = 120$ sq. in., the length of stroke $l = 16$ in., the steam being "cut off" at $\frac{1}{4}$ stroke ($\therefore a = 4$ in., and $l : a = 4.00$), and the exhaust pressure corresponds to a "vacuum of 25 inches" (by which is meant that the pressure of the exhaust steam will balance 5 inches of mercury), whence $q = \frac{5}{30}$ of $14.7 = 2.45$ lbs. per sq. inch. Required the work per stroke, W, and the corresponding power L.

Since $l : a = 4$, we have $\log_e 4 = 2.302 \times .60206 = 1.386$, and from eq. (4), (*foot*, lb., sec.,)

$$W = \tfrac{120}{144}(54.7 \times 144) \cdot \tfrac{1}{3} \cdot [2.386] - \tfrac{120}{144}(2.45 \times 144) \cdot \tfrac{4}{3}$$

$= 5165.86 - 392.0 = 4773.868$ ft. lbs. of work per stroke, and therefore the power at 2 rev. per sec. (eq. 5) is

$$L = 2 \times 2 \times 4773.87 = 19095.5 \text{ ft. lbs. per second.}$$

Hence in horse-powers, which, in ft.,-lb.-sec. system, $= L \div 550$,

$$Power = 19095.5 \div 550 = 34.7 \text{ H. P.}$$

EXAMPLE 2.—Required the weight of steam consumed per second by the above engine with given data; assuming with Weisbach that the heaviness of saturated steam at a definite pressure (and a *corresponding temperature*, § 469) is about $\frac{5}{8}$ of that of air at the same pressure and temperature.

The heaviness of air at 54.7 lbs. per sq. in. tension and temperature 287° Fahr. (see table, § 469) would be, from eq. (12) of § 472 (see also § 409),

$$\gamma = \frac{\gamma_0 T_0}{T} \cdot \frac{p}{p_0} = \frac{.0807 \times 492}{460 + 287} \cdot \frac{54.7}{14.7} = 0.198$$

lbs. per cub. foot, $\frac{5}{8}$ of which is 0.1237 lbs. per cub. ft. Now the volume* of steam, of this heaviness, admitted from the boiler at each stroke is $V = Fa = \frac{120}{144} \cdot \frac{1}{3} = 0.2777$ cub. ft., and therefore the weight of steam used per second is

$$4 \times .2777 \times 0.1237 = 0.1374 \text{ lbs.}$$

Hence, per hour, $0.1374 \times 3600 = 494.6$ lbs. of feed-water are needed for the boiler.

If, with this same engine, the steam is used at full boiler pressure throughout the whole stroke, the power will be greater, viz. $= 2nFl(p_b - q) = 33440$ ft. lbs. per sec., but the consumption* of steam will be four times as great; and hence in economy of operation it will be only 0.44 as efficient (nearly).

480. Graphic Representation of any Change of State of a Confined Mass of Gas.—The curve of expansion AB in Fig. 527 is an equilateral hyperbola, the axes X and Y being its asymptotes. If compressed air were used instead of steam its expansion curve would also be an equilateral hyperbola if its temperature could be kept from falling during the expansion (by injecting hot-water spray, e.g.), and then, following Mariotte's law, we would have, as for steam, (§ 475,) $pV =$ constant, i.e., $pFx =$ constant, and therefore $px =$ constant, which is the equation of a hyperbola, p being the ordinate and x the abscissa. This curve (dealing with a perfect gas) is also called an *isothermal*, the x and y co-ordinates of its points being proportional to the volume and tension, respectively, of a mass of air (or perfect gas) whose temperature is maintained constant.

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* We here neglect the practical fact that a portion of the fresh steam entering the cylinder is condensed prematurely, so that the actual consumption is somewhat greater than as here derived.

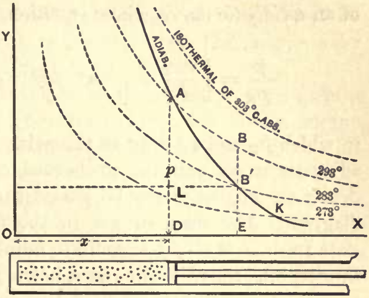

FIG 528.

Hence, *in general*, if a mass of gas be confined in a rigid cylinder of cross-section F (area), provided with an air-tight piston, Fig. 528, its volume, Fx, is proportional to the distance $OD = x$ (of the piston from the closed end of the cylinder) taken as an abscissa, while its tension p at the same instant may be laid off as an ordinate from D. Thus a point A is fixed. Describe an equilateral hyperbola through A, asymptotic to X and Y, and mark it with the observed temperature (absolute) of the air at this instant. In a similar way the diagram can be filled up with a great number of equilateral hyperbolas, or *isothermal curves*, each for its own temperature. Any point whatever (i.e., above the critical temperature) in the plane angular space YOX will indicate by its co-ordinates a volume and a tension, while the corresponding absolute temperature T will be shown by the hyperbola passing through the point, since these three variables always satisfy the relation (§ 472)

$$\frac{pV}{T} = \text{const.}; \text{ i.e., } \frac{pFx}{T} = \frac{p_0 V_0}{T_0}. \quad . \quad . \quad . \quad (1)'$$

Any change of state of the gas in the cylinder may now be represented by a line in the diagram connecting the two points corresponding to its initial and final states. Thus, a point moving along the line AB, a portion of the isothermal marked 293° Abs. Cent., represents a motion of the piston from D to E, and a consequent increase of volume, accompanied by just sufficient absorption of heat by the gas (from other bodies) to maintain its temperature at that figure (viz., its temperature at

A). If the piston move from *D* to *E*, *without transmission of heat*, i.e., *adiabatically* (§ 478), the tension falls more rapidly, and a point moving along the line AB' represents the corresponding continuous change of state. AB' is a portion of an *adiabatic curve*, whose equation, from § 478, is

$$\frac{p}{p_K} = \left[\frac{Fx_K}{Fx}\right]^{\frac{3}{2}}, \quad \text{or} \quad px^{\frac{3}{2}} = p_K x_K^{\frac{3}{2}} = \text{const.}; \quad . \quad (1)$$

in which p_K and x_K refer to the point K where this particular adiabatic curve cuts the isothermal of freezing-point. Evidently an adiabatic may be passed through any point of the diagram. The mass of gas in the cylinder may change its state from A to B' by an infinite number of routes, or lines on the diagram, the adiabatic route, however, being that most likely to occur for a rapid motion of the piston. For example, we may cool it without allowing the piston to move (and hence without altering its volume nor the abscissa x) until the pressure falls to a value $p_{B'} = DL = EB'$, and this change is represented by the vertical path from A to L; and then allow it to expand, and push the piston from D to E (i.e., do external work), during which expansion heat is to be supplied at just such a rate as to keep the tension constant, $= p_{B'} = p_L$, this latter change corresponding to the horizontal path LB' from L to B'.

It is further noticeable that the *work done* by the expanding gas upon the *near face* of the piston (or done *upon* the gas when compressed) when the space dx is described by the piston, is $= Fpdx$, and therefore is proportional to the area pdx of the small vertical strip lying between the axis X and the line or route showing the change of state; whence the total work done on the near piston-face, being $= F\int pdx$, is represented by the area $\int pdx$ of the plane figure between the initial and final ordinates, the axis X and the particular *route* followed between the initial and final states (N.B. We take no account here of the pressure on the other side of the piston, the latter depending on the style of engine). For example, the work done on the near face of the piston during adiabatic expansion from D to E is represented by the plane figure $AB'EDA$, and is measured by its area.

The mathematical relations between the quantities of heat imparted or rejected by conduction and radiation, and transformed into work, in the various changes of which the confined gas is capable, belong to the subject of *Thermodynamics*, which cannot be entered upon here.

It is now evident how the cycle of changes which a definite mass of air or gas experiences when used in a hot-air engine, compressed-air engine, or air-compressor, is rendered more intelligible by the aid of such a diagram as Fig. 528; but it must be remembered that during the entrance into, or exit from, the cylinder, of the mass of gas used in one stroke, the distance x does not represent its volume, and hence the locus of the points in the diagram determined by the co-ordinates p and x during entrance and exit does not indicate changes of state in the way just explained for the mass when confined in the cylinder. However, the *work done* by or upon the gas during entrance and exit will still be represented by the plane figure included by that locus (usually a straight horizontal line, pressure constant) and the axis of X and the terminal ordinates.

481. Adiabatic Expansion in an Engine using Compressed Air. —Fig. 529. Let the compressed air at a tension p_m and an absolute temperature T_m be supplied from a reservoir (in which the loss is continually made good by an air-compressor). Neglecting the resistance of the port, its tension and temperature when behind the piston are still p_m and T_m. Let x_n = length of stroke, and let the cut-off (or closing of communication with the reservoir) be made at some point D where $x = x_m$, the position of D being so chosen (i.e., the ratio $x_m : x_n$ so computed) that after adiabatic expansion from D to E the pressure shall have fallen from p_m at M (*state m*) to a value $p_n = p_a$

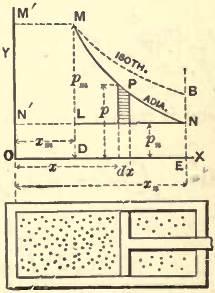

FIG. 529.

= one atmosphere at *N*(*state n*), at the end of stroke; so that when the piston returns the air will be expelled ("exhausted") at a tension equal to that of the external atmosphere (though at a low temperature). Hence the back-pressure at all points either way will be $= p_n$ per unit area of piston, and hence the total back-pressure $= Fp_n$, F being the piston area.

From O to D the forward pressure is constant and $= Fp_m$, and the effective work, therefore, or work on piston-rod from O to D, is

$$\textit{Work of entrance} = \Big[_O^D W = F[p_m - p_n]x_m, \quad . \quad . \quad (1)$$

represented by the rectangle $M'MLN'$. The cut-off being made at D, the volume of gas now in the cylinder, viz., $V_m = Fx_m$, is left to expand. Assuming no device adopted (such as injecting hot-water spray) for preventing the cooling and rapid decrease of tension during expansion, the latter is *adiabatic*, and hence the tension at any point P between M and N will be

$$p = p_m\left(\frac{x_m}{x}\right)^{\frac{3}{2}} \quad . \quad . \quad [\text{see } \S\, 478;\ V = Fx]; \quad . \quad . \quad (a)$$

$\therefore$ *Work of expansion*

$$= \Big[_D^E W = \int_{x_m}^{x_n} F(p - p_n)dx = F\int_{x_m}^{x_n} pdx - Fp_n(x_n - x_m), \quad (2)$$

and is represented by the area $MPNL$.

$$\text{But} \quad \int_m^n pdx = p_m x_m^{\frac{3}{2}} \int_{x_m}^{x_n} x^{-\frac{3}{2}}\,dx = -\,2p_m x_m^{\frac{3}{2}}\left[\left(\frac{1}{x_n}\right)^{\frac{1}{2}} - \left(\frac{1}{x_m}\right)^{\frac{1}{2}}\right];$$

$$\text{i.e.,}\ F\int_m^n pdx = 2Fp_m x_m\left[1 - \left(\frac{x_m}{x_n}\right)^{\frac{1}{2}}\right]. \quad . \quad . \quad (3)$$

Now substitute (3) in (2) and then add (2) to (1), noting that

$$F(p_m - p_n)x_m - Fp_n(x_n - x_m) = Fp_m x_m\left[1 - \frac{x_n p_n}{x_m p_m}\right],$$

which furthermore, since n and m are adiabatically related [see (a)], can be reduced to

$$Fp_m x_m\left[1 - \left(\frac{x_m}{x_n}\right)^{\frac{1}{3}}\right],$$

and we have finally:

$$\left.\begin{array}{l}\textit{Total work on piston-}\\ \quad\textit{rod per stroke}\end{array}\right\} = W = 3Fx_m p_m\left[1 - \left(\frac{x_m}{x_n}\right)^{\frac{1}{3}}\right]. \quad . \quad (4)$$

But $Fx_m = V_m$, and the adiabatic relation holds good,

$$\left(\frac{V_m}{V_n}\right)^{\frac{1}{3}} = \left(\frac{p_n}{p_m}\right)^{\frac{1}{4}};\ \text{i.e.,}\ \left(\frac{x_m}{x_n}\right)^{\frac{1}{3}} = \left(\frac{p_n}{p_m}\right)^{\frac{1}{4}};$$

therefore we may also write

$$W = 3\,V_m p_m\left[1 - \left(\frac{p_n}{p_m}\right)^{\frac{1}{4}}\right]; \quad . \quad . \quad . \quad . \quad (5)$$

in which $V_m =$ the volume which the mass of air used per stroke occupies in the *state m*, i.e., in the reservoir, where the tension is p_m and the absolute temperature $= T_m$.

To find the *work done per pound of air used* (or other unit of weight), we must divide W by the weight $G = V_m\gamma_m$ of the air used per stroke, remembering (eq. (13), § 472) that

$$V_m\gamma_m = [\,V_m p_m \gamma_0 T_0] \div (T_m p_0).$$

$$\left.\begin{array}{l}\textit{Work per unit of weight of}\\ \textit{air used in adiabatic working}\end{array}\right\} = 3T_m\frac{p_0}{\gamma_0 T_0}\left[1 - \left(\frac{p_n}{p_m}\right)^{\frac{1}{4}}\right]. \ (6)$$

The back-pressure $p_n = p_a =$ one atmosphere.

In (6) $\gamma_0 = .0807$ lbs. per cub. foot, $p_0 = 14.7$ lbs. per sq. inch, and $T = 273°$ Abs. Cent. or $492°$ Abs. Fahr.

It is noticeable in (6) that for given tensions p_m and p_n, the work per unit of weight of air used is *proportional to the absolute temperature T_m of the reservoir.* The temperature T_n to which the air has cooled at the end of the stroke is obtained as in Example 2, § 478, and may be far below freezing-point unless T_m is very high or the ratio of expansion, $x_m : x_n$, large.

EXAMPLE.—Let the cylinder of a compressed-air engine have a section of $F = 108$ sq. in. and a stroke $x_n = 15$ inches. The compressed air entering the cylinder is at a tension of 2 atmos. (i.e., $p_m = 29.4$ lbs. per sq. in., and $p_n \div p_m = \frac{1}{2}$), and at a temperature of 27° Cent. (i.e., $T_m = 300°$ Abs. Cent.). Required the proper point of cut-off, or $x_m = ?$, in order that the tension may fall to one atmosphere at the end of the stroke; also the work per stroke, and the work per pound of air. Use the *foot*, *pound*, and *second*.

From eq. (a), above, we have

$$x_m = x_n \left(\frac{p_n}{p_m}\right)^{\frac{2}{3}} = 1.25 \sqrt[3]{\frac{1}{4}} = 0.7875 \text{ ft.} = 9.45 \text{ inches,}$$

and hence the volume of air in state m, used per stroke [eq. (5)] is

$$V_m = Fx_m = \tfrac{108}{144} \times 0.7875 = 0.5906 \text{ cubic feet;}$$

while the work per stroke is

$$W = 3 \times 0.5906 \times 29.4 \times 144 \times \left[1 - (\tfrac{1}{2})^{\frac{1}{3}}\right] = 1545 \text{ ft. lbs.,}$$

and the work obtained from each pound of air, eq. (6),

$$= 3 \times 300 \times \frac{14.7 \times 144}{0.0807 \times 273} \times \left[1 - \sqrt[3]{\frac{1}{2}}\right] = 17810$$

ft. lbs. per pound of air used.

The temperature to which the air has cooled at the end of stroke [eq. (2), § 478] is

$$T_n = T_m \sqrt[3]{\frac{p_n}{p_m}} = 300 \times \sqrt[3]{\tfrac{1}{2}} = 300 \times .794 = 238^\circ \text{ Abs. C.};$$

i.e., -35° Centigrade.

482. Remarks on the Preceding.—This low temperature is objectionable, causing, as it does, the formation and gradual accumulation of snow, from the watery vapor usually found in small quantities in the air, and the ultimate blocking of the ports. By giving a high value to T_m, however, i.e., by heating the reservoir, T_n will be correspondingly higher, and also *the work per pound of air*, eq. (6). If the cylinder be encased in a "jacket" of hot water, or if spray of hot water be injected behind the piston during expansion, the temperature may be maintained nearly constant, in which event Mariotte's law will hold for the expansion, and more work will be obtained per pound of air; but the point of cut-off must be differently placed. Thus if, in eq. (4), § 479, we make the back-pressure, which $= (Fa \div Fl)p_b$, equal to the value to which the air-pressure has fallen at the end of the stroke by Mariotte's law, we have

$$\left.\begin{matrix}\textit{Work per stroke with}\\ \textit{isotherm. expans.}\end{matrix}\right\} = Fap_b \log._e \left(\frac{l}{a}\right) = V_b p_b \log._e \left(\frac{l}{a}\right), \quad (1)$$

and hence

$$\left.\begin{matrix}\textit{Work per unit of weight of air,}\\ \textit{with isothermal expansion}\end{matrix}\right\} = T_m \frac{p_0}{\gamma_0 T_0} \log._e \left(\frac{l}{a}\right). \quad . \quad (2)$$

Applying these equations to the data of the example, we obtain

$$\left.\begin{matrix}\text{Work per unit of weight of air with \textit{iso-}}\\ \textit{thermal expansion}\end{matrix}\right\} = 0.69\, T_m \frac{p_0}{\gamma_0 T_0};$$

$$\left.\begin{matrix}\text{whereas, with \textit{adiabatic expansion}, work}\\ \text{per unit of weight of air is only}\end{matrix}\right\} = 0.62\, T_m \frac{p_0}{\gamma_0 T_0}.$$

483. Double-acting Air-compressor, with Adiabatic Compression.—This is the converse of § 481. In Fig. 530 we have the piston moving from right to left, compressing a mass of air which at the beginning of the stroke fills the cylinder. This is brought about by means of an external motor (steam-engine or turbine, e.g.) which exerts a thrust or pull along the piston-rod, enabling it with the help of the atmospheric pressure of the fresh supply of air flowing in behind it, to first *compress* a cylinder-full of air to the tension of the compressed air in the reservoir, and then, the port or valve opening at this stage, to force or *deliver* it into the reservoir. Let the temperature and tension of the cylinder-full of fresh air be T_{n_1} and p_{n_1}, and the tension in the reservoir be p_{m_1}. Suppose the compression adiabatic. As the piston passes from E toward the left, the air on the left has no escape and is compressed, its tension and temperature increasing adiabatically until it reaches a value $p_{m_1} =$ that in reservoir, at which instant, the piston being at some point D, a valve opens and the further progress of the piston simply transfers the compressed air into the reservoir without further increasing its tension. Throughout the whole stroke the piston-rod has the help of one atmosphere pressure on the right face, since a new supply of air is entering on the right to be compressed in its turn on the return stroke. The work done from E to D may be called the *work of compression;* that from D to O. the *work of delivery.*

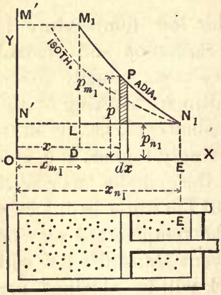

FIG. 530.

[Since, here, dx and dW (or increment of work) have contrary signs, we introduce the negative sign as shown.]

$$\textit{The work of compression} = -\int_E^D F(p - p_{n_1})dx. \quad . \quad . \quad . \quad (1c)$$

$$\textit{The work of delivery} \quad = -\int_D^O F(p_{m_1} - p_{n_1})dx. \quad . \quad . \quad (1d)$$

In these equations only p and x are variables. In the summation indicated in (1c) p changes adiabatically; in (1d) p is constant $= p_{m_1}$ as now written.

In the adiabatic compression the air passes from the state n_1 to the state m_1 (see N_1 and M_1 in figure).

The summations in these equations being of the same form as those in equations (1) and (2) of § 481, but with limits inverted, we may write immediately,

$$\textit{Work per stroke} = W = 3\,V_{m_1}p_{m_1}\left[1 - \left(\frac{p_{n_1}}{p_{m_1}}\right)^{\frac{1}{3}}\right]\;.\;.\quad (2)$$

and

$$\left.\begin{array}{l}\textit{Work per unit of weight}\\ \textit{of air compressed}\end{array}\right\} = 3T_{m_1}\frac{p_0}{\gamma_0 T_0}\left[1 - \left(\frac{p_{n_1}}{p_{m_1}}\right)^{\frac{1}{3}}\right]\quad .\quad (3)$$

The value of T_{m_1}, at the immediate end of the sudden compression, by eq. (2) of § (478), is

$$T_{m_1} = T_{n_1}\left(\frac{p_{m_1}}{p_{n_1}}\right)^{\frac{1}{3}}.\quad .\;.\;.\;.\;.\;.\quad (4)$$

The temperature of the reservoir being T_m, as in § 481 (usually much less than T_{m_1}), the compressed air entering it cools down gradually to that temperature, T_m, contracting in volume correspondingly since it remains at the same tension p_{m_1}. The mechanical equivalent of this heat is lost.

Let us now inquire what is the *efficiency of the combination* of air-compressor and compressed-air engine, the former supplying air for the latter, both working adiabatically, assuming that no tension is lost by the compressed air in passing along the reservoir between, i.e., that $p_{m_1} = p_m$. Also assume (as already implied, in fact) that $p_{n_1} = p_n$ = one atmos., and that the temperature, T_{n_1}, of the air entering the compressor cylinder is equal to that, T_m, of the reservoir and transmission-pipe.

To do this we need only find the ratio of the amount of work obtained from one pound (or other unit of weight) in the compressed-air engine to the amount spent in compressing one pound of air in the compressor. Calling this ratio η, the *efficiency*, and dividing eq. (6) of § 481 by eq. (3) of this paragraph, we have, with substitutions just mentioned,

$$\eta = \frac{T_m}{T_{m_1}} = \frac{\textit{Abs. temp. of outer free air}}{\left\{\begin{array}{l}\textit{Abs. temp. of air at end}\\ \textit{of sudden compression,}\end{array}\right.}; \quad . \quad . \quad (5)$$

or, substituting from eq. (4), and remembering that $T_{n_1} = T_m$, we have also

$$\eta = \left(\frac{p_n}{p_m}\right)^{\frac{1}{3}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

also, since

$$\left(\frac{p_n}{p_m}\right)^{\frac{1}{3}} = \frac{T_n}{T_m},$$

we may write

$$\eta = \frac{T_n}{T_m} = \frac{\text{Ab. tem. air leaving eng. cyl.}}{\text{Ab. tem. outer free air.}} \quad . \quad . \quad (7)$$

For practical details of the construction and working of engines and compressors, and the actual efficiency realized, the student may consult special works, as they lie somewhat beyond the scope of the present work.

EXAMPLE 1.—In the example of § 445, the ratio of p_m to p_n was $= \frac{1}{2}$. Hence, if compressed air is supplied to the reservoir under above conditions, the efficiency of the system is, from eq. (6), $\eta = \sqrt[3]{\frac{1}{2}} = 0.794$, about 80 per cent.

EXAMPLE 2.—If the ratio of the tensions is as small as $\frac{p_n}{p_m} = \frac{1}{6}$, the efficiency would be only $(\frac{1}{6})^{\frac{1}{3}} = 0.55$; i.e., 45 per cent of the energy spent in the compressor is lost in heat.

EXAMPLE 3.—What horse-power is required in a blowing engine to furnish 10 lbs. of air per minute at a pressure of 4 atmos., with adiabatic compression, the air being received by the compressor at one atmosphere tension and 27° Cent. (ft.-lb.-sec. system). Since 27° C. $= 300°$ Abs. C. $= T_{n_1}$, we have, from eq. (4),

$$T_{m_1} = 300\,(\tfrac{4}{1})^{\frac{1}{3}} = 477° \text{ Abs. Cent.};$$

and hence, eq. (3),

$$\textit{The work per pound of air} = 3 \times 477 \frac{14.7 \times 144}{.0807 \times 273}\left[1 - \left(\frac{1}{4}\right)^{\frac{1}{3}}\right]$$

= 50870 ft. lbs. per pound of air. Hence 10 lbs. of air will require 508700 ft. lbs. of work; and if this is done every minute we have the req. H. P. = $\frac{508700}{33000}$ = 15.4 H. P.

NOTE.—If the compression could be made *isothermal*, an approximation to which is obtained by injecting a spray of cold water, we would have, from eqs. (1) and (2) of § 482:

$$\left.\begin{matrix}\textit{Work per}\\ \textit{lb. air}\end{matrix}\right\} = T_{n_1}\frac{p_0}{\gamma_0 T_0}\log._e\left(\frac{p_{m_1}}{p_{n_1}}\right) = \frac{300 \times 14.7 \times 144}{.0807 \times 273} \times 1.386$$

= 39950 ft. lbs. per lb., and the corresponding H. P. = 12.1; a saving of about 25 per cent, compared with the former. The difference was employed in heating the air in the air-compressor with adiabatic compression, and was lost when that extra heat was dissipated in the reservoir as the air cooled again. This difference is easily shown graphically by comparing in the same diagram the areas representing the work done in the two cases.

484. Hot-air Engines.—Since we have seen that the tension of air and other gases can be increased by heating, if the volume be kept the same, a mass of air thus treated can afterwards be allowed to expand in a working cylinder, and thus become a means of converting heat into work. In *Stirling's* hot-air engine a definite confined mass of air is used indefinitely without loss (except that occasional small supplies are needed to make up for leakage), and is alternately heated and cooled. A displacement-plunger, or piston, fitting loosely in a bell-like chamber, is so connected with the piston of the working cylinder and the fly-wheel, that its forward stroke is made while the other piston waits at the beginning of its stroke. In this motion the plunger causes the confined air to pass in a thin sheet over the top and sides of the furnace dome, thus greatly increasing its tension. The air then expands behind the working piston with falling tension and temperature, and, while that piston pauses at the end of its forward stroke, is again shifted in position, though without change of volume, by the return stroke of the plunger, in such a way as to pass through a coil of pipes in which cold water is flowing. This reduces both its temperature and tension, and hence its resistance to the piston on the return stroke is at first less than atmospheric, but is gradually increased by the compression. This cycle of changes is repeated indefinitely, and is easily traced on a diagram like that in Fig. 528, and computations made accordingly.

A special invention of Stirling's is the "*regenerator*" or box filled with numerous sheets of wire gauze, in its passage through which the working air, after expansion, deposits some of its heat, which it *re-absorbs* to some extent when, after further cooling in the "refrigerator" or pipe coil and compression by the return stroke of the piston, it is made to pass backward through the regenerator to be further heated by the furnace in readiness for a forward stroke. This feature, however, has not realized all the expectations of its inventor and improvers, as to economy of heat and fuel.

In *Ericsson's* hot-air engine, of more recent date, the displacement-plunger fits its cylinder air-tight, but valves can be opened through its edges when moving in one direction, thus causing it to act temporarily as a loose plunger, or shifter. The two pistons move simultaneously in the same direction in the same cylinder, but through different lengths of stroke, so that the space between them is alternately enlarged and contracted. The working piston also has valves opening through it for receiving a fresh supply of air into the space between the two pistons. During the forward stroke a fresh instalment from the outer air enters through the working piston into the space between it and the other, whose valves are now closed and which is now expelling from its further face, through proper valves, the air used in the preceding stroke; no work is done in this stroke. On the return stroke this fresh supply of air is free to expand behind the now retreating working piston, while its tension is greatly increased by its being shifted (at least a large portion of it) over the furnace dome through the valves (now open) of the plunger piston, by the motion of the latter, which now acts as a loose plunger. The engine is therefore only single-acting, no work being done in each forward stroke. For further details, see Goodeve's and Rankine's works on the steam-engine; also the article "Hot-air Engine" in Johnson's Cyclopædia by Pres. Barnard, and Röntgen's Thermodynamics.

485. Brayton's Petroleum-engine.—Although a more recent invention than the gas-engines to be mentioned presently, this motor is more closely related to hot-air engines than the latter. By a *slow* combustion of petroleum vapor the gaseous products of combustion, while under considerable tension, are enabled to follow up a piston with a sustained pressure. being left to expand through the latter part of the stroke. Thus we have the furnace and working cylinder combined in one. The gradual combustion is accomplished by making use of the principle of the Davy safety-lamp that flame will not spread through layers of wire gauze of proper fineness.

486. Gas-engines.—We again have the furnace and working cylinder in one in a "*gas-engine*," where illuminating gas and atmospheric air are introduced into the working cylinder in proper proportions (about ten parts of air to one of gas, by weight) to form an explosive mixture of more or less violence and exploded at a certain point of the stroke, causing a very sudden rise of temperature and tension, after which the mass expands behind the piston with falling pressure. On the return stroke the products of combustion are expelled, and no work done, these engines being single-acting. In some forms the mixture is compressed before explosion, since it has been found that under this treatment a mixture containing a larger proportion of air to gas can be made to ignite, and that then the resulting pressure is more gradual and sustained, like that of steam or of the mixture in the Brayton engine. That is, the effect is analogous to that of "slow-burning powder" in a gun.

In the "Otto Silent Gas-engine" the explosion occurs only every fourth stroke, and one side of the piston is always open to the air. The action on the other side of the piston is as follows: (1) In the forward stroke a fresh supply of explosive mixture is drawn into the cylinder at one atmosphere tension. (2) The next (backward) stroke compresses the mixture into about one fourth of its original bulk, this operation occurring at the expense of the kinetic energy of the fly-wheel. (3) The mixture is ignited, the pressure rises to 8 or 10 atmospheres, and work is done on the piston through the next (forward) stroke, the tension of the products of combustion having fallen to about two atmospheres at the end of the stroke. (4) In the next (backward) stroke the products of combustion are expelled and no work is done.

The Atkinson "Cycle Gas-engine," an English invention of recent date (see the *London Engineer* for May 1887; pp. 361 and 380) also makes an explosion every fourth stroke, but the link work connecting the piston and fly-wheel is of such design that the latter makes but one revolution during the four strokes. Also the length of the expansion or working stroke is greater than that of the compression stroke and the products of combustion are *completely expelled.* Consequently the efficiency of this motor is at present greater than that of any other gas-engine. See § 487.

One of the most simple gas-engines is made by the Economic Motor Company of New York. The piston has no packing, being a long plunger ground to fit the cylinder accurately and kept well lubricated. As with most gas-engines the cylinder is encased in a water-jacket to prevent excessive heating of the working parts and consequent decomposition of the lubricant.

For further details on these motors, see Rankine's "Steam-engine," Clark's "Gas-engines" in Van Nostrand's Science Series, and article "Gas-engine" in Johnson's Cyclopædia; also Prof. Thurston's report on Mechanical Engineering at the Vienna Exhibition of 1873, and proceedings of the "Society of Engineers" (England) for 1881.

487. Efficiency of Heat-engines.—According to the mechanical theory of heat, the combustion of one pound of coal, producing, as it does, about 14,000 heat-units (British Thermal units; see § 149, Mechanics) should furnish

$$14{,}000 \times 772 = 10{,}808{,}000 \text{ ft. lbs. of work,}$$

if entirely converted into work. Let us see how nearly this is accomplished in the performance of the most recent and economical marine engines of the present day, viz., the triple expansion engines of some Atlantic steamers, which are claimed to have consumed per hour only 1.25 lbs. of coal for each measured ("indicated") horse-power of effective work done in their cylinders. The work-equivalent of 1.25 lbs. of coal per hour is

$$1.25 \times 14{,}000 \times 772 = 13{,}510{,}000 \text{ ft. lbs. per hour;}$$

while the actual work per hour implied in "one H. P. per hour" is

$$33000 \times 60 = 1{,}980{,}000 \text{ ft. lbs. per hour.}$$

That is, the engines utilize only *one seventh* of the heat of combustion of the fuel.

According to Prof. Thurston, this is a rather extravagant claim (1.25), the actual consumption having probably been 1.4 lbs. of coal per H. P. per hour.

The ordinary compound marine engine is stated to use as little as 2.00 lbs. per hour for each H. P.

Most of the heat not utilized is dissipated in the condenser.

Similarly, the water-jacket, a necessary evil in the operation of the gas-engine, is a source of great loss of heat and work. Still, Mr. Wm. Anderson in his recent work, "Conversion of Heat into Work" (London, 1887), mentions a motor of this class as having converted into work $\frac{1}{6}$ of the heat of combustion [an Otto "Silent Gas-engine," tested at the Stevens Institute, Hoboken, N. J., in 1883]; while Prof. Unwin found the Atkinson engine (see last paragraph) capable of returning (in the cylinder) fully $\frac{1}{5}$ of the heat-equivalent of the gas consumed. [This latter result was confirmed in Philadelphia in Jan. 1889 by Prof. Barr, under direction of Prof. Thurston.]

488. Duty of Pumping-engines.—Another way (often used in speaking of the performance of pumping-engines) of expressing the degree of economy attained in the use of fuel by the combined furnace, boiler, and engine is to give the number of foot-pounds of work obtained from each 100 lbs. of coal consumed in the furnace, calling it the "*duty*" of the engine.

For example, by a duty of 99,000,000 ft. lbs. it is meant that from each pound of coal 990,000 ft. lbs. of work are obtained. From this we gather that, since one horse-power consists of $33,000 \times 60 = 1,980,000$ ft. lbs. per hour, the engine mentioned must use each hour

$1,980,000 \div 990,000 = 2$ lbs. of coal for each H. P. developed;

which is as low a figure as that attained by the marine engines last quoted.

489. Buoyant Effort of the Atmosphere.—In the case of a body of large bulk but of small specific gravity the buoyant effort of the air (due to the same cause as that of water, see § 456) becomes quite appreciable, and may sometimes be greater than the weight of the body. This buoyant effort is equal to the weight of air displaced, i.e., $= V\gamma$, where V is the volume of air displaced, and γ its heaviness.

If G_1 = total weight of the body producing the displacement, the resultant vertical force is

$$P = G_1 - V\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for equilibrium, or *suspension in the air*, we must have $P = 0$, i.e.,

$$G_1 = V\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We may therefore find approximately the elevation where a given balloon will cease to ascend, by determining the heaviness γ of the air at that elevation from eq. (2); then, knowing approximately the temperature of the air at that elevation, we may compute its tension p [eq. (13), § 472], and finally, from eqs. (3), (4), or (5) of § 477, obtain the altitude required.

EXAMPLE.—The car and other solid parts of a balloon weigh

400 lbs., and the bag contains 12,000 cub. feet of illuminating gas weighing 0.030 lb. per cub. foot at a tension of one atmosphere and temperature of 15° Cent., so that its total weight $= 12{,}000 \times 0.030 = 360$ lbs.

Hence $G_1 = 760$ lbs. We may also write with sufficient accuracy: Whole volume of displacement $= V = 12{,}000$ cub. ft.

As the balloon ascends the exterior pressure diminishes, and the confined gas tends to expand and so increase the volume of displacement V; but this we shall suppose prevented by the strength of the envelope. At the surface of the ground (station n of Fig. 531; see also Fig. 526) let the barometer read 29.6 inches and the temperature be 15° Cent. Then $T_n = 288°$ Abs. Cent., and the heaviness of the air at n is

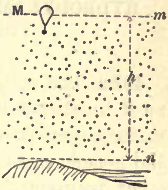

FIG. 531.

$$\gamma_n = \frac{.0807 \times 273}{14.7} \cdot \frac{\frac{29.6}{30} \times 14.7}{288}$$

$$\left(= \frac{\gamma_0 T_0 p_n}{p_0 T_n}\right) = .0807 \times \frac{273}{288} \cdot \frac{29.6}{30} = .0754 \text{ lbs. per cub. ft.}$$

At the unknown height h, where the balloon is to come to rest, i.e., at M, G_1 must $= V\gamma$ [eq. (2)]; therefore

$$\gamma_m = \frac{G_1}{V} = \frac{760 \text{ lbs.}}{12{,}000 \text{ cub. ft.}} = .0633 \text{ lbs. per cub. ft.};$$

and if the temperature at M be estimated to be 5° Cent. (or $T_m = 278°$ Abs. Cent.) (on a calm day the temperature decreases about 1° Cent. for each 500 ft. of ascent), we shall have, from $\frac{p_m}{\gamma_m T_m} = \frac{p_n}{\gamma_n T_n}$,

$$\frac{p_n}{p_m} = \frac{\gamma_n T_n}{\gamma_m T_m} = \frac{.0754}{.0633} \cdot \frac{288}{278} = 1.206;$$

and hence, from eq. (5), § 477, with $\frac{1}{2}(T_m + T_n)$ put for T_n,

$$h = 26213 \times \tfrac{283}{273} \times 2.30258 \times \log._{10} 1.206 = 5088 \text{ ft.}$$

CHAPTER VI.

HYDRODYNAMICS BEGUN—STEADY FLOW OF LIQUIDS THROUGH PIPES AND ORIFICES.

489a. The subject of **Water in Motion** presents one of the most unsatisfactory branches of Applied Mechanics, from a mathematical stand-point. The internal eddies, cross-currents, and general intricacy of motion of the particles among each other, occurring in a pipe transmitting a fluid, are almost entirely defiant of mathematical expression, though the flow of water through a circular orifice in a thin plate into the air presents a simpler case, where the conception of "stream lines" is probably quite close to the truth. In most practical cases we are forced to adopt as a basis for mathematical investigation the simple assumption that the particles move side by side in such a way that those which at any instant form a lamina or thin sheet, ⅂ to the axis of the pipe or orifice, remain together as a lamina during the further stages of the flow. This is the Hypothesis of Flow in Plane Layers, or Laminated Flow. Experiment is then relied on to make good the discrepancies between the indications of the formulæ resulting from this theory and the actual results of practice; so that the science of Hydrodynamics is largely one of coefficients determined by experiment.

490. Experimental Phenomena of a "Steady Flow."—As preliminary to the analysis on which the formulæ of this chapter are based, and to acquire familiarity with the quantities involved, it will be advantageous to study the phenomena of the apparatus represented in Fig. 532. A large tank or reservoir BC is connected with another, DE, at a lower level, by means of a rigid pipe opening under the water-level in each tank. This pipe has no sharp curves or bends, is of various sectional areas at different parts, the changes of section being very gradual, and the highest point N_2 not being more than 30 ft. higher than BC, the surface-level of the upper tank. Let both tanks be filled with water (or other liquid), which will also rise to H and to K in the pipe. Stop the ends L and N_4 of the pipe, and through M, a stop-cock in the highest curve, pour in water to fill the remainder of the pipe; then, closing M, unstop L and N_4.

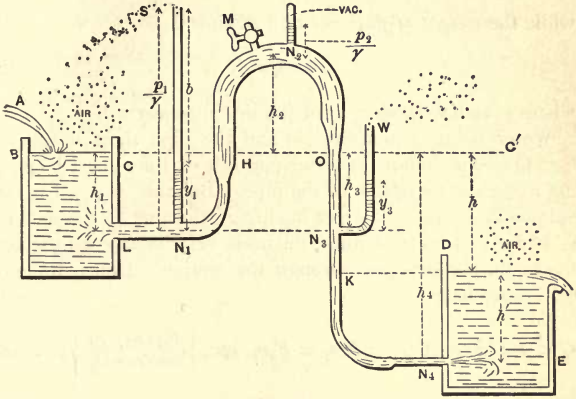

FIG. 532.

If the dimensions are not extreme (and subsequent formulæ will furnish the means of testing such points) the water will now begin to flow from the upper tank into the lower, and all parts of the pipe will continue full of water as the flow goes on.

Further, suppose the upper tank *so large that its surface-level sinks very slowly;* or that an influx at A continually makes good the efflux at E; then the flow is said to be a **Steady Flow**; or, *a state of permanency* is said to exist; i.e., the circumstances of the flow at each section of the pipe are *permanent*, or *steady*.

By measuring the volume, V, of water discharged at E in a time t, we obtain the *volume of flow per unit of time*, viz.,

$$Q = \frac{V}{t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while the *weight of flow* per unit of time is

$$G = Q\gamma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\gamma =$ heaviness (§ 7) of the liquid concerned.

Water being incompressible and the pipe rigid, it follows that the same volume of water per unit of time must be passing at each cross-section of the pipe. But this is equal to the volume of a prism of water having F, the area of the section, as a base, and, as an altitude, the mean velocity $= v$ with which the liquid particles pass through the section. Hence for any section we have

$$Q = Fv = \text{constant} = F_1v_1 = F_2v_2, \text{ etc.} \left\{ \begin{matrix} \textit{Equat. of} \\ \textit{continuity} \end{matrix} \right\}, \quad . \quad (3)$$

in which the subscripts refer to different sections. If the flow were *unsteady*, e.g., if the level BC were sinking, this would be true for a definite instant of time; but when *steady*, we see that it is permanently true; e.g., F_1v_1 at any instant $= F_2v_2$ at the same or *any other* instant, subsequent or previous. In other words, *in a steady flow the velocity at a given section remains unchanged with lapse of time.*

[N.B. We here assume for simplicity that the different particles of water passing simultaneously through a given section (i.e., abreast of each other) have equal velocities, viz., the velocity which all other particles will assume on reaching this section. Strictly, however, the particles at the sides are somewhat retarded by friction on the surface of the pipe. This assumption is called the *Assumption of Parallel Flow*, or *Flow in Plane Layers*, or *Laminated Flow.*]

Let us suppose Q to have been found as already prescribed. We may then, knowing the internal sectional areas at different parts of the pipe, N_1, N_2, etc., compute the velocities

$$v_1 = Q \div F_1, \quad v_2 = Q \div F_2, \quad \text{etc.},$$

which the water must have in passing those sections, respectively. It is thus seen that the velocity at any section has no direct connection with the height or depth of the section from the plane, BC, of the upper reservoir surface. The fraction $\frac{v^2}{2g}$ will be called the *height due to the velocity*, v, or simply the *velocity-head*, for convenience.

Next, as to the value of the internal fluid pressure, p, per unit-area (in the water itself and against the side or wall of pipe) at different sections of the pipe. If the end N_4 of the pipe were stopped, the problem would be one in Hydrostatics, and the pressure against the side of the pipe at N_1 (also at N_3 on same level) would be simply

$$p_1 = p_a + h_1\gamma,$$

measured by a water column of height

$$\frac{p_1}{\gamma} = \frac{p_a}{\gamma} + h_1 = b + h_1,$$

in which p_a = one atmosphere, and b = 34 ft. = height of an ideal water barometer, and γ = 62.5 lbs. per cubic foot; and this would be shown experimentally by screwing into the side of the pipe at N_1 a small tube open at both ends; the water would rise in it to the level BC. That is, a column of water of height = h_1 would be sustained in it, which indicates that the internal pressure at N_1 corresponds to an ideal water column of a height

$$= \frac{p_1}{\gamma} = b + h_1.$$

But when a steady flow is proceeding, the case being now one of Hydrodynamics, we find the column of water sustained at rest in the small tube (called an *open piezometer*) N_1S has a height y_1, *less* than h_1, and hence the internal fluid pressure is less than it was when there was no flow. This pressure being called p_1, the ideal water column measuring it has a height

$$\frac{p_1}{\gamma} = b + y_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

at N_1, and will be called the *pressure-head* at the section referred to. We also find experimentally that while the flow is steady the piezometer-height y (and therefore the pressure-head $b + y$) at any section remains unchanged with lapse of time, as a characteristic of a steady flow.

[For correct indications, the extremity of the piezometer should have its edges flush with the inner face of the pipe wall, where it is inserted.]

At N_3, although at the same level as N_1, we find, on inserting an open piezometer, W, that with $F_3 = F_1$ (and therefore with $v_3 = v_1$) y_3 is a little less than y_1; while if $F_3 < F_1$ (so that $v_3 > v_1$), y_3 is not only less than y_1, but the difference is greater than before. We have therefore found experimentally that, in a general way, when water is flowing in a pipe it presses less against the side of the pipe than it did before the flow was permitted, or (what amounts to the same thing) the pressure between the transverse laminæ is less than the hydrostatic pressure would be.

In the portion HN_2O of the pipe we find the pressure less than one atmosphere, and consequently a manometer registering pressures from zero upward (and not simply the excess over one atmosphere, like the Bourdon steam-gauge and the open piezometer just mentioned) must be employed. At N_2, e.g., we find the pressure

$$= \tfrac{1}{2} \text{ atmos., i.e., } \frac{p_2}{\gamma} = 17 \text{ ft.}$$

Even below the level BC, by making the sections quite narrow (and consequently the velocities great) the pressure may be made less than one atmosphere. At the surface BC the pressure is of course just one atmosphere, while that in the jet at N_4, entering the right-hand tank under water, is necessarily

$p_4 = 1$ atmos. + press. due to col. h' of water practically at rest;

$$\text{i.e., } \frac{p_4}{\gamma} = \text{pressure-head at } N_4 = b + h';$$

(whereas if N_4 were stopped by a diaphragm, the pressure-head just on the right of the diaphragm would be $b + h'$, and that on the left $b + h_4$.)

Similarly, when a jet enters the atmosphere in parallel filaments its particles are under a pressure of one atmosphere, i.e., their pressure-head $= b = 34$ ft. (for water); for the air immediately around the jet may be considered as a pipe between which and the water is exerted a pressure of one atmosphere.

491. Recapitulation and Examples.—We have found experimentally, then, that in a steady flow of liquid through a rigid pipe there is at each section of the pipe a definite velocity and pressure which all the liquid particles assume on reaching that section; in other words, at each section of the pipe the liquid velocity and pressure remain constant with progress of time.

EXAMPLE 1.—If in Fig. 532, the flow having become steady, the volume of water flowing in 3 minutes is found on measurement to be 134 cub. feet, the volume per second is, from eq. (1), § 490,

$$Q = \tfrac{134}{180} = 0.744 \text{ cub. ft. per second.}$$

EXAMPLE 2.—If the flow in 2 min. 20 sec. is 386.4 lbs., the volume of flow per second is [ft., lb., sec.; eqs. (1) and (2)]

$$Q = \frac{V}{t} = \frac{G}{\gamma} \div t = \frac{386.4}{62.5} \cdot \frac{1}{140} = 0.0441 \text{ cub. ft. per sec.}$$

EXAMPLE 3.—In Fig. 532 the height of the open piezometer at N_1 is $y_1 = 9$ feet; what is the internal fluid pressure? [Use the inch, lb., and sec.] The internal pressure is

$$p_1 = p_a + y_1\gamma = 14.7 + 108 \times \tfrac{62.5}{1728} = 18.6 \text{ lbs. per sq. inch.}$$

The pressure on the outside of the pipe is, of course, one atmosphere, so that the resultant bursting pressure at that point (N_1) is 3.9 lbs. per sq. in.

EXAMPLE 4.—The volume of flow per second being .0441 cub. ft. per sec., as in Example 1, required the velocity at a section of the (circular) pipe where the diameter is 2 inches. [Use ft., lb., and sec.]

$$v = \frac{Q}{F} = \frac{0.0441}{\frac{1}{4}\pi(\frac{2}{12})^2} = 2.02 \text{ ft. per sec.};$$

while at another section of the pipe where the diameter is four inches (double the former) and the sectional area, F, is therefore four times as great, the velocity is $\frac{1}{4}$ of $2.02 = 0.505$ ft. per sec.

492. Bernoulli's Theorem for Steady Flow; without Friction.—If the pipe is comparatively short, without sudden bends, elbows, or abrupt changes of cross-section, the effect of friction of the liquid particles against the sides of the pipe and against each other (as when eddies are produced, disturbing the parallelism of flow) is small, and will be neglected in the present analysis, whose chief object is to establish a formula for steady flow through a short pipe and through orifices.

An assumption, now to be made, of *flow in plane layers*, or *laminated flow*, i.e., flow in laminæ ⌉ to the axis of the pipe at every point, may be thus stated: (see Fig. 533, which shows a steady flow proceeding, through a pipe CD of indefinite extent.) All the liquid particles which at any instant form a small lamina, or sheet, as AB, ⌉ to axis of pipe, *keep company as a lamina throughout the whole flow.* The thickness, ds', of this lamina remains constant so long as the pipe is of constant cross-section, but shortens up (as at C) on passing through a larger section, and lengthens out (as at D) in a part of the pipe where the section is smaller (i.e., the sectional area, F, is smaller). The mass of such a lamina is $Fds'\gamma \div g$ [§ 55], its velocity at any section will be called v (pertaining to that point of the pipe's axis), the pressure of the lamina just behind it is Fp, upon the rear face, while the resistance (at the same instant) offered by its neighbor just ahead is $F(p + dp)$ on the front face; also

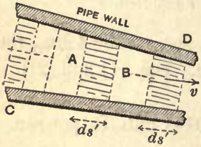

FIG. 533.

its weight is the vertical force $Fds'\gamma$. Fig. 534 shows, as a free body, the lamina which at any instant is passing a point A of the pipe's axis, where the velocity is v and pressure p.

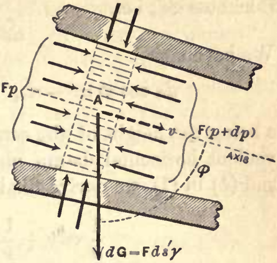

FIG. 534.

Note well the forces acting; the pressures of the pipe wall on the edges of the lamina have no components in the direction of v, for the wall is considered smooth, i.e., those pressures are ˥ to wall; in other words, no friction is considered. To this free body apply eq. (7) of § 74, for any instant of any curvilinear motion of a material point

$$vdv = (tang.\ acceleration) \times ds, \quad . \quad . \quad . \quad . \quad (1)$$

in which ds = a small portion of the path, and is described in the time dt. Now the tang. accel. = Σ(tang. compons. of the acting forces) ÷ mass of lamina, i.e.,

$$tang.\ acc. = \frac{Fp - F(p + dp) + F\gamma ds' \cos \phi}{F\gamma ds' \div g}. \quad . \quad (2)$$

Now, Fig. 535, at a *definite instant of time*, conceive the volume of water in the pipe to be subdivided into a great number of laminæ of *equal mass* (which implies equal volumes

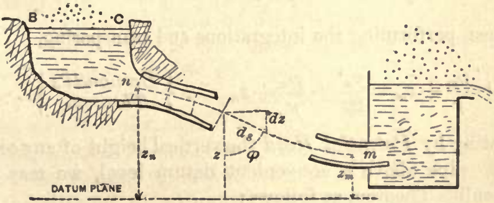

FIG. 535.

in the case of a liquid, but not with gaseous fluids), and let the ds just mentioned for any one lamina be the distance from its centre to that of the one next ahead; this mode of subdivision makes the *ds* of any one lamina identical in value with its thickness *ds'*, i.e.,

$$ds = ds'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have also

$$ds \cos \phi = - dz, \quad \text{or} \quad ds' \cos \phi = - dz; \quad . \quad . \quad (4)$$

z being the height of the centre of a lamina above any convenient horizontal datum plane. Substituting from (2), (3), and (4) in (1), we derive finally

$$\frac{1}{g} vdv + \frac{1}{\gamma} dp + dz = 0. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The flow being steady, and the subdivision into laminæ being of the nature just stated, each lamina in some small time *dt* moves into the position which at the beginning of *dt* was filled by the lamina next ahead, *and acquires the same velocity, the same pressures on its faces, and the same value of z, that the front lamina had at the beginning of dt.*

Hence, considering the simultaneous advance made by all the laminæ in this same *dt*, we may write out an equation like (5) for each of the laminæ between any two cross-sections *n* and *m* of the pipe, thus obtaining an infinite number of equations, from which by adding corresponding terms, i.e., *by integration*, we obtain

$$\frac{1}{g}\int_{v_n}^{v_m} vdv + \frac{1}{\gamma}\int_{p_n}^{p_m} dp + \int_{z_u}^{z_m} dz = 0; \quad . \quad . \quad . \quad (6)$$

whence, performing the integrations and transposing,

$$\frac{{v_m}^2}{2g} + \frac{p_m}{\gamma} + z_m = \frac{{v_n}^2}{2g} + \frac{p_n}{\gamma} + z_n \quad . \quad . \left\{ \begin{matrix} \textit{Bernoulli's} \\ \textit{Theorem} \end{matrix} \right\} \quad . \quad . \quad (7)$$

Denoting by *Potential Head* the vertical height of any section of the pipe above a convenient datum level, we may state Bernoulli's Theorem as follows:

In steady flow without friction, the sum of the velocity-head, pressure-head, and potential head at any section of the pipe is a constant quantity, being equal to the sum of the corresponding heads at any other section.

It is noticeable that in eq. (7) each of the terms is a linear quantity, viz., a height, or head, either actual, such as z_n and z_m, or ideal (all the others), and does not bring into account the absolute size of the pipe, nor even its relative dimensions (v_m and v_n, however, are connected by the equation of continuity $F_m v_m = F_n v_n$), and contains no reference to the volume of water flowing per unit of time [Q] or the shape of the pipe's axis. When the pipe is of considerable length compared with its diameter the friction of the water on the sides of the pipe cannot be neglected (§ 512).

It must be remembered that Bernoulli's Theorem does not hold unless the flow is *steady*, i.e., unless each lamina, in coming into the *position* just vacated by the one next ahead (of equal mass), comes also into the exact conditions of velocity and pressure in which the other was when in that position.

[N.B. This theorem can also be proved by applying to all the water particles between n and m, as a collection of small *rigid* bodies (water being incompressible) the theorem of Work and Energy for a collection of Rigid Bodies in § 142, eq. (xvi), taking the respective paths which they describe simultaneously in a single dt.]

493. First Application of Bernoulli's Theorem without Friction. —Fig. 536 shows a large tank from which a vertical pipe of uniform section leads to another tank and dips below the surface of the water in the latter. Both surfaces are open to the air. The vessels and pipe being filled with water, and the lower end m of the pipe unstopped, a steady flow is established almost immediately, the surface BC being very large compared with F, the area of the (*uniform*) section of the pipe.

Given F, and the heights h_0 and h, required the velocity v_m of the jet at m and also the pressure, p_n, at n (in pipe near entrance of same). m is in the jet, just clear of the pipe, and practically in the water-level, AD. The velocity v_m is unknown, but the pressure p_m is practically $= p_a =$ one atmosphere, since

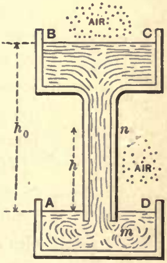

FIG. 536.

the pressure on the sides of the jet is necessarily the hydrostatic pressure due to a slight depth below the surface AD.

$\therefore$ *Press.-head at m is* $\dfrac{p_m}{\gamma} = \dfrac{p_a}{\gamma} = b = 34$ feet. . . (§ 423)

Now apply Bernoulli's Theorem to sections m and n, taking a horizontal plane through m as a datum plane for potential heads, so that $z_n = h$ and $z_m = 0$, and we have

$$\frac{v_m^2}{2g} + b + 0 = \frac{v_n^2}{2g} + \frac{p_n}{\gamma} + h. \quad . \quad . \quad . \quad . \quad (1)$$

But, *assuming that the section of the pipe is filled at every point*, we must have

$$v_m = v_n;$$

for, in the equation of continuity

$$F_m v_m = F_n v_n,$$

if we put $F_m = F_n$, the pipe being of uniform section, we obtain $v_m = v_n$. Hence eq. (1) reduces to

$$\frac{p_n}{\gamma} = b - h = 34 \text{ ft.} - h. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the pressure at n is *less than one atmosphere*, and if a small tube communicating with an air-tight receiver full of air were screwed into a small hole at n, the air in the receiver would gradually be drawn off until its tension had fallen to a value p_n. [This is the principle of *Sprengel's air-pump*, mercury, however, being used instead of water, as for this heavy liquid $b =$ only 30 inches.]

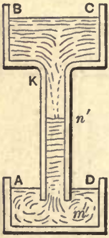

FIG. 537.

If h is made $> b$ for water, i.e. > 34 feet (or > 30 inches for mercury), p_n would be negative from eq. (2), which is impossible, showing that the assumption of full pipe-sections is not borne out. In this case, $h > b$, only a portion, mn', (in length somewhat less than b,) of the tube will be kept full during the flow (Fig. 537); while in the part Kn' vapor of water, of low tension corresponding to the temperature (§ 469), will surround an internal jet which does not fill the pipe. As for the value of v_m, Bernoulli's Theorem, applied to BC and m, in Fig. 536, gives finally $v_m = \sqrt{2gh_0}$.

Example.—If $h = 20$ feet, Fig. 536, and the liquid is water, the pressure-head at n is (ft., lb., sec.)

$$\frac{p_n}{\gamma} = b - h = 34' - 20' = 14 \text{ ft.},$$

and therefore

$$p_n = 14 \times 62.5 = 875 \text{ lbs. per sq. ft.} = 6.07 \text{ lbs. per sq. in.}$$

494. Second Application of Bernoulli's Theorem without Friction.—Knowing by actual measurement the open piezometer height y_n at the section n in Fig. 538 (so that the pressure-head, $\frac{p_n}{\gamma} = b + y_n$, at n is known); knowing also the vertical distance h_n from n to m, and the respective cross-sections F_n and F_m (F_m being the sectional area of the jet, flowing into the air, so that $\frac{p_m}{\gamma} = b$), required the volume of flow per sec.; i.e., required Q, which

$$= F_n v_n = F_m v_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

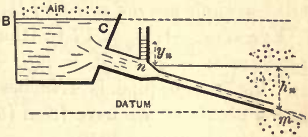

Fig. 538.

The pipe is short, with smooth curves, if any, and friction will therefore be neglected. From Bernoulli's Theorem [eq. (7), § 492], taking m as a datum plane for potential heads, we have

$$\frac{v_m^2}{2g} + b + 0 = \frac{v_n^2}{2g} + (y_n + b) + h_n. \quad . \quad . \quad . \quad (2)$$

But from (1) we have

$$v_n = [F_m \div F_n]v_m;$$

substituting which in (2) we obtain, solving for v_m,

$$v_m = \frac{\sqrt{2g(y_n + h_n)}}{\sqrt{1 - \left(\frac{F_m}{F_n}\right)^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and hence the volume per unit of time becomes known, viz.,

$$Q = F_m v_m . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

NOTE.—If the cross-section F_m of the nozzle, or jet, is $> F_n$, v_m becomes imaginary (unless y_n is *negative* (i.e., $p_n <$ one atmos.), and numerically $> h_n$); in other words, the assigned cross-sections are *not filled by the flow.*

EXAMPLE.—If $y_n = 17$ ft. (thus showing the internal fluid pressure at n to be $p_n = \gamma(y_n + b) = 1\frac{1}{2}$ atmos.), $h_n = 10$ ft., and the (round) pipe is 4 inches in diameter at n and 3 inches at the nozzle m, we have from (3) (using ft.-lb.-sec. system of units in which $g = 32.2$)

$$v_m = \frac{\sqrt{2 \times 32.2(17 + 10)}}{\sqrt{1 - \left[\frac{\frac{1}{4}\pi 3^2}{\frac{1}{4}\pi 4^2}\right]^2}} = 50.4 \text{ ft. per sec.}$$

[N.B. Since $F_m \div F_n$ is a ratio and therefore an abstract number, the use of the inch in the ratio will give the same result as that of the foot.]

Hence, from (4),

$$Q = F_m v_m = \tfrac{1}{4}\pi(\tfrac{3}{12})^2 \times 50.4 = 2.474 \text{ cub. ft. per sec.}$$

495. Orifices in Thin Plate.—Fig. 539. When efflux takes place through an orifice in a thin plate, i.e., a *sharp-edged* orifice in the plane wall of a tank, a contracted vein (or " vena contracta") is formed, the filaments of water not becoming parallel until reaching a plane, m, parallel to the plane of vessel wall, which for circular orifices is at a distance from the interior plane of vessel wall equal to the radius of the circular aperture; and not until reaching this plane does the internal fluid pressure become equal to that of the surrounding medium (atmosphere, here), i.e., surrounding the jet. We assume that the width of the orifice is small compared with h, unless the vessel wall is horizontal.

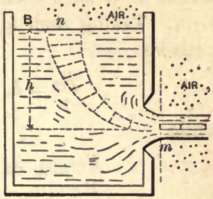

FIG. 539.

The area of the cross-section of the jet at m, called the *contracted section*, is found on measurement to be from .60 to .64 of the area of the aperture with most orifices of ordinary shapes, even with widely different values of the area of aperture and of the height, or head, h, producing the flow. Calling this abstract number [.60 to .64] the *Coefficient of Contraction*, and denoting it by C, we may write

$$F_m = CF,$$

in which $F =$ area of the orifice, and $F_m =$ that of the contracted section. C ranges from .60 to .64 with circular orifices, but may have lower values with some rectangular forms. (See table in § 503.)

A lamina of particles of water is under atmospheric pressure at n (the free surface of the water in tank or reservoir), while its velocity at n is practically zero, i.e. $v_n = 0$ (the surface at B being very large compared with the area of orifice). It experiences increasing pressure as it slowly descends until in the immediate neighborhood of the orifice, when its velocity is rapidly accelerated and pressure decreased, in accordance with Bernoulli's Theorem, and its shape lengthened out, until finally at m it forms a portion of a filament of a jet, its pressure is one atmosphere, and its velocity, $= v_m$, we wish to determine. The course of this lamina we call a "*stream-line*," and Bernoulli's Theorem is applicable to it, just as if it were enclosed in a frictionless pipe of the same form. Taking then a datum plane through the centre of m, we have

$$\frac{p_m}{\gamma} = b, \quad z_m = 0, \quad \text{and} \quad v_m = ?;$$

while

$$\frac{p_n}{\gamma} \text{ also } = b, \quad z_n = h, \quad \text{and} \quad v_n = 0.$$

Hence Bernoulli's Theorem gives

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h;$$

$$\therefore \frac{v_m^2}{2g} = h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$v_m = \sqrt{2gh}.$$

That is, *the velocity of the jet at m is theoretically the same as that acquired by a body falling freely in vacuo through a height $= h =$* the "head of water." We should therefore expect that if the jet were directly vertically upward, as at m, Fig. 540, a height $\frac{v_m^2}{2g}$ would be actually attained. [See §§ 52 and 53.] Experiment shows that the height of the jet (at m) does not materially differ from h if h is not $>$ 6 or 8 feet. For $h >$ 8 ft., however, the actual height reached is $< h$, the difference being not only absolutely but relatively greater as h is taken greater, since the resistance of the air is then more and more effective in depressing and breaking up the stream. (See § 578.)

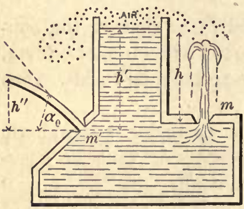

FIG. 540.

At m', Fig. 540, we have a jet, under a head $= h'$, directed at an angle α_0 with the horizontal. Its form is a parabola (§ 81), and the theoretical height reached is $h'' = h' \sin^2 \alpha_0$ (§ 80).

The jet from an orifice in thin plate is very limpid and clear. From eq. (1), we have theoretically

$$v_m = \sqrt{2gh}$$

(an equation we shall always use for efflux *into the air* through *orifices* and *short pipes* in the plane wall of a large tank whose water-surface is very large compared with the orifice, and is open to the air), but experiment shows that for an "*orifice in thin plate*" this value is reduced about 3% by friction at the edges, so that for ordinary practical purposes we may write

$$v_m = \phi \sqrt{2gh} = 0.97 \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad (2)$$

in which ϕ is called the *coefficient of velocity.*

Hence the volume of flow, Q, per time-unit will be

$$Q = F_m v_m = CF\phi \sqrt{2gh}, \ \textit{on the average} = 0.62F \sqrt{2gh}. \quad (3)$$

It is to be understood that the flow is steady, and that the reservoir surface (very large) and the jet are *both under atmospheric pressure.* ϕC is called the *coefficient of efflux.*

EXAMPLE 1.—Fig. 539. Required the velocity of efflux, v_m, at m, and the volume of the flow per second, Q, into the air, if $h = 21$ ft. 6 inches, the *circular* orifice being 2 in. in diam.; take $C = 0.64$. [Ft., lb., and sec.]

From eq. (2),

$$v_m = 0.97 \sqrt{2 \times 32.3 \times 21.5} = 36.1 \text{ ft. per sec.};$$

hence the discharge is

$$Q = F_m v_m = 0.64 \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 \times 36.1 = 0.504 \text{ cub. ft. per second.}$$

EXAMPLE 2.—[Weisbach.] Under a head of 3.396 metres the velocity v_m in the contracted section is found by measurements of the jet-curve to be 7.98 metres per sec., and the discharge proves to be 0.01825 cub. metres per sec. Required the coefficient of velocity (ϕ) and that of contraction (C), if the area of the orifice is 36.3 sq. centimetres.

Use the *metre-kilogram-second* system of units, in which $g = 9.81$ met. per sq. second.

From eq. (2),

$$\phi = \frac{v_m}{\sqrt{2gh}} = \frac{7.98}{\sqrt{2 \times 9.81 \times 3.396}} = 0.978;$$

while from (3) we have

$$C = \frac{Q}{F\phi\sqrt{2gh}} = \frac{Q}{Fv_m} = \frac{.01825}{\frac{363}{10000} \times 7.98} = 0.631.$$

ϕ and C, being abstract numbers, are independent of the system of concrete units adopted.

NOTE.—To find the velocity v_m of the jet at the orifice by measurements of the jet-curve, as mentioned in Example 2, we may proceed as follows: Since we cannot very readily assure ourselves that the direction of the jet at the orifice is horizontal, we consider the angle α_0 of the parabola (see Fig. 93 and § 80) as unknown, and therefore have two unknowns to deal with, and obtain the necessary two equations by measuring the x and y (see page 84) of two points of the jet, remembering that if we use the equation (3) of page 84 in its present form points of the jet below the orifice will have negative y's. The substitution of these values x_1, x_2, y_1, and y_2 in equation (3) furnishes two equations between constants, in which only α_0 and h are unknown. To eliminate α_0, for $\frac{1}{\cos^2 \alpha_0}$ we write $1 + \tan^2 \alpha_0$, and taking $x_2 = 2x_1$ for convenience, we finally obtain

$$h = \frac{1}{8} \cdot \frac{x_2^2 + (4y_1 - y_2)^2}{y_2 - 2y_1}, \text{ and } \therefore v_m = \sqrt{\frac{g[x_2^2 + (4y_1 - y_2)^2]}{4(y_2 - 2y_1)}},$$

in which y_1 and y_2 are the vertical distances of the two points chosen *below* the orifice; that is, we have already made them negative in eq. (3) of page 84. The h of the preceding equation simply denotes $v_m^2 \div 2g$, and must not be confused with that of the last two figures. For accuracy the second point should be as far from the orifice along the jet as possible.

496. Orifice with Rounded Approach.*—Fig. 541 shows the general form and proportions of an orifice or mouth-piece in the use of which contraction does not take place beyond the edges, the inner surface being one "of revolution," and so shaped that the liquid filaments are parallel on passing the outer edge m; hence the pressure-head at m is $= b$ ($= 34$ ft. for water and 30 inches for mercury) in Bernoulli's Theorem, if efflux takes place *into the air.* We have also the sectional area $F_m = F =$ that of final edge of orifice, i.e., the coefficient of contraction, or C, $=$ unity $= 1.00$, so that the discharge per time-unit has a volume

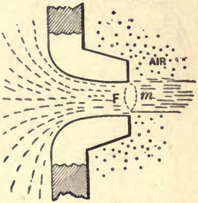

Fig. 541.

$$Q = F_m v_m = F v_m.$$

The tank being large, as in Fig. 540, Bernoulli's Theorem applied to m and n will give, as before,

$$v_m = \sqrt{2gh}$$

as a theoretical result, while practically we write

$$v_m = \phi \sqrt{2gh}, \quad \ldots \ldots \ldots \quad (1)$$

and

$$Q = F\phi \sqrt{2gh}. \quad \ldots \ldots \ldots \quad (2)$$

As an average ϕ is found to differ little from 0.97 with this orifice, the same value as for an orifice in thin plate (§ 495).

497. Problems in Efflux Solved by Applying Bernoulli's Theorem.—In the two preceding paragraphs the pressure-heads at sections m and n were each $= p_a \div \gamma =$ height of

---

[*] Smooth conical nozzles for fire-steams give $\phi = .97$ with $h =$ press.-head + veloc.-head at base of play-pipe; see p. 833.

the liquid barometer $= b$; but in the following problems this will not be the case necessarily. However, efflux is to take place through a simple orifice in the side of a large reservoir, whose upper surface (n) is very large, so that v_n may be put $=$ zero.

Problem I.—Fig. 542. What is the velocity of efflux, v_m, at the orifice m (i.e., at the contracted section, if it is an orifice in thin plate) of a jet of water from a steam-boiler, if the free surface at n is at a height $= h$ above m, and the pressure of the steam over the water is p_n, the discharge taking place into the air?

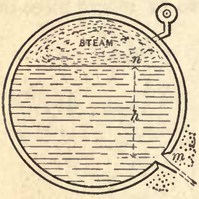

FIG. 542.

Applying Bernoulli's Theorem to section m at the orifice [where the pressure-head is b and velocity-head $v_m^2 \div 2g$ (unknown)] and to section n at water-surface (where velocity-head $= 0$ and pressure-head $= p_n \div \gamma$), we have, taking m as a datum for potential heads so that $z_m = 0$ and $z_n = h$,

$$\frac{v_m^2}{2g} + b + 0 = 0 + \frac{p_n}{\gamma} + h;$$

$$\therefore\ v_m = \sqrt{2g\left[\frac{p_n}{\gamma} - b + h\right]}. \quad . \quad . \quad . \quad . \quad (1)$$

EXAMPLE.—Let the steam-gauge read 40 lbs. (and hence $p_n = 54.7$ lbs. per sq. inch) and $h = 2$ ft. 4 in.; required v_m.

Also if $F = \frac{1}{2}$ sq. in., in "thin plate," required the rate of discharge (volume). The temperature of saturated steam of the given tension must be 286° Fahr. [see foot of page 607]. The water is practically at the same temperature and hence of a heaviness, γ, of 57.7 lbs. per cubic ft. (p. 518).

From eq. (1) above, then, with ft. lb. and sec., noting that for this case $b = [(14.7 \times 144) \div 57.7]$ feet,

$$v_m = \sqrt{2 \times 32.2\left[\frac{54.7 \times 144}{57.7} - \frac{14.7 \times 144}{57.7} + \frac{28}{12}\right]}$$

$= 81.1$ ft. per sec., theoretically; but practically

$$v_m = 0.97 \times 81.1 = 78.6 \text{ ft. per sec.};$$

so that the discharge begins at the rate of

$$Q = 0.64\,Fv_m = 0.64 \times \tfrac{1}{2} \cdot \tfrac{1}{144} \times 78.6 = 0.174 \text{ cub. ft. p. sec.}$$

Problem II.—Fig. 543. With what velocity, v_m, will water flow into the condenser C of a steam-engine where the tension of the vapor is p_m, < one atmosphere, if h = the head of water, and the flow takes place through an orifice in thin plate? Taking position m in the contracted section where the filaments are parallel, and the pressure therefore equal to that of the surrounding vapor, viz., p_m, and position n in the (wide) free surface of the water in the tank, where (at surface) the pressure is one atmosphere [and $\therefore \frac{p_n}{\gamma} = b = 34$ ft.] and velocity practically zero; we have, applying Bernoulli's Theorem to n and m, taking m as a datum level for potential heads (so that $z_n = h$ and $z_m = 0$),

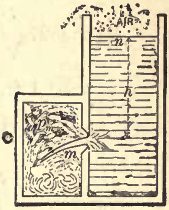

FIG. 543.

$$\frac{v_m^2}{2g} + \frac{p_m}{\gamma} + 0 = 0 + b + h,$$

$\therefore$

$$v_m = \sqrt{2g\left[h + b - \frac{p_m}{\gamma}\right]}, \quad . \quad . \quad . \quad . \quad (1)$$

and

$$Q = F_m v_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as theoretical results. But practically we must write

$$v_m = 0.97\sqrt{2g\left[h + b - \frac{p_m}{\gamma}\right]}, \quad . \quad . \quad . \quad . \quad (3)$$

and

$$Q = F_m v_m = CFv_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which F = area of orifice in thin plate, and C = coefficient of contraction = about 0.62 approximately [see § 495].

EXAMPLE.—If in the condenser there is a "vacuum" of $27\frac{1}{2}$ inches (meaning that the tension of the vapor would support $2\frac{1}{2}$ inches of mercury, in a barometer), so that

$$p_m = [\tfrac{2.5}{30} \times 14.7] \text{ lbs. per sq. inch, and } h = 12 \text{ feet,}$$

while the orifice is $\frac{1}{2}$ inch in diameter; we have, using the ft., lb., and sec.,

$$v_m = 0.97\sqrt{2 \times 32.2\left[12 + 34 - \frac{\frac{5}{60} \times 14.7 \times 144}{62.5}\right]}$$

$$= 51.1 \text{ ft. per sec.}$$

(We might also have written, for brevity,

$$\frac{p_m}{\gamma} = [2\tfrac{1}{2} : 30] \times 34 = 2.833,$$

since the pressure-head for one atmos. = 34 feet, for water. Hence, for a circular orifice in thin plate, we have the volume discharged per unit of time,

$$Q = CFv = 0.62 \times \frac{\pi}{4}\left(\frac{\frac{1}{2}}{12}\right)^2 \times 51.1 = 0.0431 \text{ cub. ft. per sec.}$$

497a. Efflux through an Orifice in Terms of the Internal and External Pressures.—Fig. 544. Let efflux take place through a small orifice from the plane side of a large tank, in which at the level of the orifice the *hydrostatic pressure* was $= p'$ before the opening of the orifice, that of the medium surrounding the jet being $= p''$. When a steady flow is established, after opening the orifice, the pressure in the water on a level with the orifice will not be materially changed, *except in the immediate neighborhood of the orifice* [see § 495]; hence, applying Bernoulli's Theorem to m in the jet, where the filaments are parallel, and a point n, in the body of the liquid and at the same level as m, and *where the particles are practically at rest* [i.e., $v_n = 0$] (hence not too near the

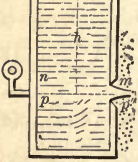

FIG. 544.

orifice), we shall have, cancelling out the potential heads which are equal,

$$\frac{v_m^2}{2g} + \frac{p''}{\gamma} = \frac{0^2}{2g} + \frac{p'}{\gamma},$$

$$\therefore\ v_m = 0.97\sqrt{2g\left[\frac{p' - p''}{\gamma}\right]}. \quad . \quad . \quad . \quad (1)$$

(In Fig. 544 p' would be equal to $p_a + h\gamma$.) Eq. (1) is conveniently applied to the jet produced by a *force-pump*, supposing, for simplicity, the orifice to be *in the head of the pump-cylinder*, as shown in Fig. 545. Let the thrust (force) exerted along the piston-rod be $= P$, and the area of the piston be $= F'$. Then the intensity of internal pressure produced in the chamber AB (when the piston moves uniformly) is

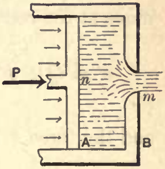

FIG. 545.

$$p' = \frac{P + F'p_a}{F'},$$

while the external pressure in the air around the jet is simply p_a (one atmos.).

$$\therefore\ v_m = 0.97\sqrt{2g \cdot \frac{P}{F'\gamma}}. \quad . \quad . \quad . \quad . \quad (1)'$$

(N.B. Of course, at points near the orifice the internal pressure is $< p'$; read § 495.)

EXAMPLE.—Let the force, or thrust, P, [due to steam-pressure on a piston not shown in figure,] be 2000 lbs., and the diameter of pump-cylinder be $d = 9$ inches, the liquid being *salt water* (so that $\gamma = 64$ lbs. per cubic foot).

Then

$$F' = \tfrac{1}{4}\pi(\tfrac{9}{12})^2 = 0.442 \text{ sq. ft.},$$

and [ft., lb., sec.]

$$v_m = 0.97\sqrt{2 \times 32.2 \times \frac{2000}{0.442 \times 64}} = 65.4 \text{ ft. per sec.}$$

If the orifice is well rounded, with a diameter of one inch, the volume discharged per second is

$$Q = F_m v_m = F v_m = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 \times 65.4 = 0.353 \text{ cub. ft. per sec.}$$

To maintain steadily this rate of discharge, the piston must move at the rate [veloc. $= v'$] of

$$v' = Q \div F' = .353 \div \left[\frac{\pi}{4}\left(\frac{9}{12}\right)^2\right] = 0.800 \text{ ft. per sec.,}$$

and the force P must exert a *power* (§ 130) of

$$L = Pv' = 2000 \times 0.800 = 1600 \text{ ft. lbs. per sec.}$$
$$= \text{about 3 horse-power (or 3 H. P.).}$$

If the water must be forced from the cylinder through a pipe or hose before passing out of a nozzle into the air, the velocity of efflux will be smaller, on account of "*fluid friction*" in the hose, for the same P; such a problem will be treated later [§ 513]. Of course, in a pumping-engine, by the use of several pump-cylinders, and of air-chambers, a practically steady flow is kept up, notwithstanding the fact that the motion of each piston is not uniform, and must be reversed at the end of each stroke.

498. Influence of Density on the Velocity of Efflux in the Last Problem.—From the equation

$$v_m = \sqrt{2g\frac{p' - p''}{\gamma}}$$

of the preceding paragraph, where p'' is the external pressure around the jet, and p' the internal pressure at the same level as the orifice but well back of it, *where the liquid is sensibly* *at rest,* we notice that for the same *difference* of pressure $[p'-p'']$ *the velocity of efflux is inversely proportional to the square root of the heaviness of the liquid.* Hence, *for the same* $(p'-p'')$, mercury would flow out of the orifice with a velocity only 0.272 of that of water; for

$$\sqrt{\frac{62.5}{848}}=\sqrt{\frac{1}{13.5}}=\frac{272}{1000}.$$

Again, assuming that the equation holds good for the flow of gases (as it does approximately when p' does not greatly exceed p''; e.g., by 6 or 8 per cent), the velocity of efflux of atmospheric air, when at a heaviness of 0.807 lbs. per cub. foot, would be

$$\sqrt{\frac{62.5}{.0807}}=\sqrt{775.3}=27.8$$

times as great as for water, with the same $p'-p''$. (See § 548, etc.)

499. Efflux under Water. Simple Orifice.—Fig. 546. Let h_1 and h_2 be the depths of the (small) orifice below the levels of the "head" and "tail" waters respectively. Then, using the formula of § 457, we have for the pressure at n (at same level as m, the jet)

$$p'=(h_1+b)\gamma,$$

and for the external pressure, around the jet at m,

$$p''=(h_2+b)\gamma;$$

whence, theoretically,

$$v_m=\sqrt{2g\frac{p'-p''}{\gamma}}=\sqrt{2g(h_1-h_2)}=\sqrt{2gh}, \quad . \quad (1)$$

where $h=$ *difference of level between the surfaces of the two bodies of water.*

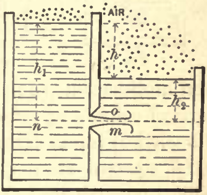

FIG. 546.

Practically, $$v_m = \phi \sqrt{2gh}\,; \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

but the value of ϕ for efflux under water is somewhat uncertain; as also that of C, the coefficient of contraction. Weisbach says that μ, $= \phi C$, is $\frac{1}{15}$ part less than for efflux into the air; others, that there is no difference (Trautwine). See also p. 389 of vol. 6, Jour. of Engin. Associations, where it is stated that with a circular mouth-piece of 0.37 in. diam., and of "nearly the form of the *vena contracta*," μ was found to be .952 for discharge into the air, and .945 for submerged discharge.

500. Efflux from a Small Orifice in a Vessel in Motion.

CASE I. *When the motion is a vertical translation and uniformly accelerated.*—Fig. 547. Suppose the vessel to move upward with a constant acceleration $\bar{p}$. (See § 49a.) Taking m and n as in the two preceding paragraphs, we know that $p_m = p''$ = external pressure = one atmos. $= p_a$ (and $\therefore \frac{p_m}{\gamma} = b$). As to the internal pressure at n (same level as m, but well back of orifice), p_n, this is not equal to $(b + h)\gamma$, because of the accelerated motion, but we may determine it by considering free the vertical column or prism On of liquid, of cross-section $= dF$, the vertical forces acting on which are $p_a dF$, downward at O, $p_n dF$ upward at n, and its weight, downward, $hdF\gamma$. All other pressures are horizontal. For a vertical upward acceleration $= \bar{p}$, the algebraic sum of the vertical components of all the forces must = *mass* × *vert. accel.*,

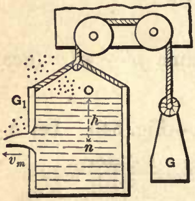

FIG. 547.

i.e., $$dF(p_n - p_a - h\gamma) = \frac{h\gamma dF}{g} \cdot \bar{p};$$

whence

$$p_n = p_a + h\gamma\left[1 + \frac{\bar{p}}{g}\right]. \quad . \; . \; . \; . \; . \; . \quad (1)$$

Putting p_n and p_a equal to the p' and p'', respectively, of the equation, we have

$$v_m = \sqrt{2g\left[\frac{p_n - p_m}{\gamma}\right]}$$

of § 497,

$$v_m = \sqrt{2(g+\bar{p})h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It must be remembered that v_m is the velocity of the jet *relatively to the orifice*, which is itself in motion with a variable velocity. The absolute velocity w_m of the particles of the jet is found by the construction in § 83, being represented graphically by the diagonal of a parallelogram one of whose sides is v_m, and the other the velocity c with which the orifice itself is moving at the instant, as part of the vessel. The jet may make any angle with the side of the vessel.

On account of the flow the internal pressures of the water against the vessel are no longer balanced horizontally, and the latter will swing out of the vertical unless properly constrained. If $\bar{p} = g =$ acc. of gravity, $v_m = \sqrt{2}\ \sqrt{2gh}$. If $\bar{p}$ is *negative* and $= g$, $v_m = 0$; i.e., there is no flow, but both the vessel and its contents fall freely, without mutual action.

CASE II. *When the liquid and the vessel both have a uniform rotary motion* about a vertical axis with an angular velocity $= \omega$ (§ 110). *Orifice small*, so that we may consider the liquid inside (except near the orifice) to be in relative equilibrium. Suppose the jet horizontal at m, Fig. 548, and the radial distance of the orifice from the axis to be $= x$. The external pressure $p_m = p_a$, and the internal [see § 428, eqs. (3) and (4)] is

$$p_n = p_a + (h_0 + z)\gamma = p_a + h_0\gamma + \frac{\omega^2 x^2}{2g}\gamma;$$

hence the velocity of the jet, relatively to the orifice, is (from § 497, since p_n and p_m correspond to the p' and p'' of that article),

$$v_m = \sqrt{2g\frac{(p_n - p_m)}{\gamma}} = \sqrt{2gh_0 + (\omega x)^2},$$

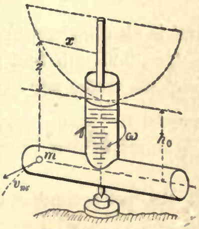

FIG. 548.

i.e., $$v_m = \sqrt{2gh_0 + w^2}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which $w, = \omega x,$ = the (constant) linear velocity of the orifice in its circular path. The *absolute velocity* w_m *of the particles in the jet* close to the orifice is the diagonal formed on w and v_m (§ 83). Hence by properly placing the orifice in the casing, w_m may be made small or large, and thus the kinetic energy carried away in the effluent water be regulated, within certain limits. Equation (3) will be utilized subsequently in the theory of Barker's Mill.

Example.—Let the casing make 100 revol. per min. (whence $\omega = [2\pi 100 \div 60]$ *radians* per sec.), $h_0 = 12$ feet, and $x = 2$ ft.; then (ft., lb., sec.)

$$v_m = \sqrt{2 \times 32.2 \times 12 + \left(\frac{2\pi 100 \times 2}{60}\right)^2} = 34.8 \text{ ft. per sec.}$$

(while, if the casing is not revolving, $v_m = \sqrt{2gh_0}$ = only 27.8 ft. per sec.).

If the jet is now directed horizontally and backward, and also tangentially to the circular path of the centre of the orifice, its *absolute velocity* (i.e., relatively to the earth) is

$$w_m = v_m - \omega x = 34.8 - 20.9 = 13.9 \text{ ft. per sec.,}$$

and is also horizontal and backwards. If the volume of flow is $Q = 0.25$ cub. feet per sec., the *kinetic energy carried away with the water per second* (§ 133) is

$$= \tfrac{1}{2}Mw_m^2 = \frac{Q\gamma}{g}\cdot\frac{w_m^2}{2} = \frac{\frac{1}{4} \times 62.5}{32.2}\,\frac{(13.9)^2}{2} = 46.8$$

ft. lbs. per second = 0.085 horse-power.

501. Theoretical Efflux through Rectangular Orifices of Considerable Vertical Depth, in a Vertical Plate.—If the orifice is so large vertically that the velocities of the different filaments in a vertical plane of the stream are theoretically different, having different "heads of water," we proceed as follows, taking into account, also, *the velocity of approach*, c, or mean velocity (if any appreciable), of the water in the channel approaching the orifice.

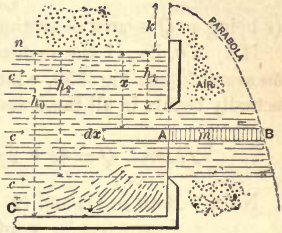

FIG 549.

Fig. 549 gives a section of the side of the tank and orifice. Let $b =$ width of the rectangle, the sills of the latter being *horizontal*, and $a = h_2 - h_1$, its height. Disregarding contraction for the present, the theoretical volume of discharge per unit of time is equal to the sum of the volumes like $v_m dF$ ($= v_m b dx$), in which $v_m =$ the velocity of any filament, as m, in the jet, and $bdx =$ cross-section of the small prism which passes through any horizontal strip of the area of orifice, in a unit of time, its altitude being v_m. For each strip there is a different x or "head of water," and hence a different velocity. Now the *theoretical* discharge (volume) per unit of time is

$$Q = \textit{sum of the volumes of the elem. prisms} = \int_{x=h_1}^{x=h_2} v_m dF;$$

i.e.,

$$Q = b\int_{h_1}^{h_2} v_m dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)'$$

But from Bernoulli's Theorem, if $k = c^2 \div 2g =$ the velocity-head at n, the surface of the *channel of approach* nC, b being the pressure-head of n, and x its potential head referred to m as datum (N.B. This $b = 34$ ft. for water, and must not be confused with the width b of orifice), we have [see § 492, eq. (7)]

$$\frac{v_m^2}{2g} + b + 0 = \frac{c^2}{2g}(\text{or } k) + b + x;$$

$$\therefore\ v_m = \sqrt{2g}\,\sqrt{x + k}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)'$$

and since $dx = d(x + k)$, k being a constant, we have, from (1)′ and (2)′,

$$\textit{Theoret. } Q = b\sqrt{2g}\int_{h_1+k}^{h_2+k} (x + k)^{\frac{1}{2}} d(x + k),$$

or

$$\text{Theoret. } Q = \tfrac{2}{3} b \sqrt{2g}\left[(h_2 + k)^{\frac{3}{2}} - (h_1 + k)^{\frac{3}{2}}\right]. \quad . \quad (1)$$

(b now denotes the width of orifice.) If c is small, the channel of approach being large, we have

$$\text{Theoret. } Q = \tfrac{2}{3} b \sqrt{2g}\,(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}) \quad . \quad . \quad . \quad . \quad (2)$$

(c being $= Q \div$ area of section of nC).

If $h_1 = 0$, i.e., if the orifice becomes a *notch in the side*, or an *overfall* [see Fig 550, which shows the contraction which actually occurs in all these cases], we have for an *overfall*

$$\text{Theoret. } Q = \tfrac{2}{3} b \sqrt{2g}\left[(h_2 + k)^{\frac{3}{2}} - k^{\frac{3}{2}}\right]. \quad . \quad . \quad . \quad . \quad (3)$$

NOTE.—Both in (1) and (2) h_1 and h_2 are the vertical depths of the respective sills of the orifice from the *surface of the water three or four feet back of the plane of the orifice*, where the surface is comparatively level. This must be specially attended to in deriving the actual discharge from the theoretical (see § 503).

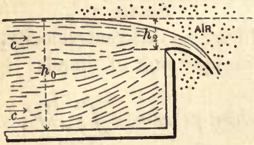

FIG. 550.

If Q were the unknown quantity in eqs. (1) and (3) it would be necessary to proceed by successive assumptions and approximations, since Q is really involved in k; for

$$k = \frac{c^2}{2g} \quad \text{and} \quad F_0 c = Q$$

(where F_0 is the sectional area of the channel of approach nC).

With $k = 0$ (or c very small, i.e., F_0 very large), eq. (3) reduces (for an *overfall*) to

$$\text{Theoret. } Q = \tfrac{2}{3} b h_2 \sqrt{2gh_2}, \quad . \quad . \quad . \quad . \quad (3\tfrac{1}{2})$$

or $\frac{2}{3}$ as much as if all parts of the orifice had the same head of water $= h_2$ (as for instance if the orifice were in the horizontal bottom of a tank in which the water was h_2 deep, the orifice having a width $= b$ and length $= h_2$).

502. Theoretical Efflux through a Triangular Orifice in a Thin Vertical Plate or Wall. Base Horizontal.—Fig. 551. Let the channel of approach be so large that the velocity of approach may be neglected. h_1 and h_2 = depths of sill and vertex, which is downward. The analysis differs from that of the preceding article only in having $k = 0$ and the length u, of a horizontal strip of the orifice, variable; b being the length of the base of the triangle. From similar triangles we have

$$\frac{u}{b} = \frac{h_2 - x}{h_2 - h_1}; \text{ i.e., } u = \frac{b}{h_2 - h_1}(h_2 - x).$$

$$\therefore \textit{Theoret. } Q = \int v_m dF = \int v_m u dx = \frac{b}{h_2 - h_1}\int_{h_1}^{h_2} v_m (h_2 - x) dx;$$

and finally, substituting from eq. (2)′ of § 501, with $k = 0$,

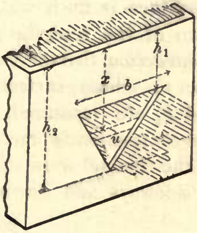

FIG. 551.

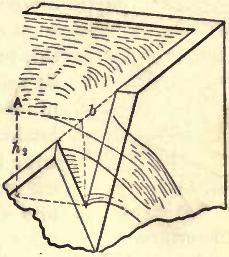

FIG. 552.

$$\textit{Theoret. } Q = \frac{b\sqrt{2g}}{h_2 - h_1}\int_{h_1}^{h_2}(h_2 - x)x^{\frac{1}{2}}dx$$

$$= \frac{2}{15}\frac{b\sqrt{2g}}{h_2 - h_1}\left[2h_2^{\frac{5}{2}} - 5h_2 h_1^{\frac{3}{2}} + 3h_1^{\frac{5}{2}}\right]. \quad . \quad . \quad . \quad (4)$$

For a *triangular notch* as in Fig. 552, this reduces to

$$\textit{Theoret. } Q = \frac{4}{15} bh_2 \sqrt{2gh_2} = \frac{8}{15}\frac{bh_2}{2}\sqrt{2gh_2}; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

i.e., $\frac{8}{15}$ of the volume that would be discharged per unit of time if the triangular orifice with base b and altitude h_2 were cut in the horizontal bottom of a tank under a head of h_2. The measurements of h_2 and b are made with reference to the *level surface* back of the orifice (see figure); for the water-surface in the plane of the orifice is curved below the level surface in the tank.

Prof. Thomson has found by experiment that with $b = 2h_2$, the actual discharge = theoret. disch. $\times$ 0.595; and with $b = 4h_2$, actual = theoret. disch. $\times$ 0.620.

503. Actual Discharge through Sharp-edged Rectangular Orifices (sills horizontal) *in the vertical side of a tank or reservoir.*

CASE I. *Complete and Perfect Contraction.*—The actual volume of water discharged per unit of time is much less than the theoretical values derived in § 501, chiefly on account of contraction. By *complete contraction* we mean that no edge of the orifice is flush with the side or bottom of the reservoir; and by *perfect contraction*, that the channel of approach, to whose surface the heads h_1 and h_2 are measured, is so large that the contraction is practically the same if the channel were of infinite extent sideways and downward from the orifice.

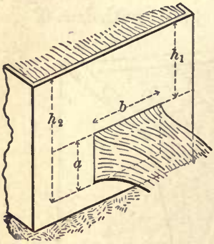

FIG. 553.

For this case (h_1 not zero) it is found most convenient to use the following practical formula (b = width):

$$\text{Actual } Q = \mu_0 ab\sqrt{2g\left[h_1 + \frac{a}{2}\right]}, \quad . \quad . \quad . \quad (6)$$

in which (see Fig. 553) a = the height of orifice, h_1 = the vertical depth of the upper edge of the orifice below the level of the reservoir surface, *measured a few feet back of the plane of the orifice*, and μ_0 is a *coefficient of efflux* (an abstract number), dependent on experiment.

With $\mu_0 = 0.62$ approximate results (within 3 or 4 per cent) may be obtained from eq. (6) with openings not more than

18 inches, or less than 1 inch, high; and not less than 1 inch wide; with heads $\left(h_1 + \frac{a}{2}\right)$ from 1 ft. to 20 or 30 feet.

Example.—What is the actual discharge (volume) per minute through the orifice in Fig. 553, 14 inches wide and 1 foot high, the upper sill being 8 ft. 6 in. below the surface of still water? Use eq. (6) with the ft., lb., and sec. as units, and $\mu_0 = 0.62$.

Solution:

$$Q = 0.62 \times 1 \times 1\tfrac{1}{6} \times \sqrt{2 \times 32.2[8\tfrac{1}{2} + \tfrac{1}{2}]} = 17.41 \text{ cub. ft. per. sec.}$$

while the *flow of weight* is

$$G = Q\gamma = 17.41 \times 62.5 = 1088 \text{ lbs. per second.}$$

Poncelet and Lesbros' Experiments.—For comparatively accurate results, values of μ_0 taken from the following table (computed from the careful experiments of Poncelet and Lesbros) may be used for the sizes there given, and, where practicable, for other sizes by interpolation. To use the table, the values of h_1, a, and b must be reduced to metres, which can be done by the reduction-table below; but in substituting in eq. (6), if the metre-kilogram-second system of units be used g must be put $= 9.81$ metres per square second (see § 51), and Q will be obtained in cubic metres per second.

Since μ_0 is an abstract number, once obtained as indicated above, it does not necessitate any particular system of units in making substitutions in eq. (6). The ft., lb., and sec. will be used in subsequent examples.

TABLE FOR REDUCING FEET AND INCHES TO METRES.

1 foot	= 0.30479 metre.	1 inch	= 0.0253 metre.
2 feet	= 0.60959 “	2 inches	= 0.0507 “
3 “	= 0.91438 “	3 “	= 0.0761 “
4 “	= 1.21918 metres.	4 “	= 0.1015 “
5 “	= 1.52397 “	5 “	= 0.1268 “
6 “	= 1.82877 “	6 “	= 0.1522 “
7 “	= 2.13356 “	7 “	= 0.1776 “
8 “	= 2.43836 “	8 “	= 0.2030 “
9 “	= 2.74315 “	9 “	= 0.2283 “
10 “	= 3.04794 “	10 “	= 0.2536 “
		11 “	= 0.2790 “

TABLE, FROM PONCELET AND LESBROS.

VALUES OF μ_0, FOR EQ (6), FOR RECTANGULAR ORIFICES IN THIN PLATE.

(Complete and perfect contraction.)

Value of h_1, Fig. 553 (in metres).	$b = .20^{m.}$ $a = .20^{m.}$	$b = .20^{m.}$ $a = .10^{m.}$	$b = .20^{m.}$ $a = .05^{m.}$	$b = .20^{m.}$ $a = .03^{m.}$	$b = .20^{m.}$ $a = .02^{m.}$	$b = .20^{m.}$ $a = .01^{m.}$	$b = .60^{m.}$ $a = .20^{m.}$	$b = .60^{m.}$ $a = .02^{m.}$
	μ_0	μ_0	μ_0	μ_0	μ_0	μ_0	μ_0	μ_0
0.005						0.705		
.010			0.607	0.630	0.660	.701		0.644
.015		0.593	.612	.632	.660	.697		.644
.020	0.572	.596	.615	.634	.659	.694		.643
.030	.578	.600	.620	.638	.659	.688	0.593	.642
.040	.582	.603	.623	.640	.658	.683	.595	.642
.050	.585	.605	.625	.640	.658	.679	.597	.641
.060	.587	.607	.627	.640	.657	.676	.599	.641
.070	.588	.609	.628	.639	.656	.673	.600	.640
.080	.589	.610	.629	.638	.656	.670	.601	.640
.090	.591	.610	.629	.637	.655	.668	.601	.639
.100	.592	.611	.630	.637	.654	.666	.602	.639
.120	.593	.612	.630	.636	.653	.663	.603	.638
.140	.595	.613	.630	.635	.651	.660	.603	.637
.160	.596	.614	.631	.634	.650	.658	.604	.637
.180	.597	.615	.630	.634	.649	.657	.605	.636
.200	.598	.615	.630	.633	.648	.655	.605	.635
.250	.599	.616	.630	.632	.646	.653	.606	.634
.300	.600	.616	.629	.632	.644	.650	.607	.633
.400	.602	.617	.628	.631	.642	.647	.607	.631
.500	.603	.617	.628	.630	.640	.644	.607	.630
.600	.604	.617	.627	.630	.638	.642	.607	.629
.700	.604	.616	.627	.629	.637	.640	.607	.628
.800	.605	.616	.627	.629	.636	.637	.606	.628
.900	.605	.615	.626	.628	.634	.635	.606	.627
1.000	.605	.615	.626	.628	.633	.632	.605	.626
1.100	.604	.614	.625	.627	.631	.629	.604	.626
1.200	.604	.614	.624	.626	.628	.626	.604	.625
1.300	.603	.613	.622	.624	.625	.622	.603	.624
1.400	.603	.612	.621	.622	.622	.618	.603	.624
1.500	.602	.611	.620	.620	.619	.615	.602	.623
1.600	.602	.611	.618	.618	.617	.613	.602	.623
1.700	.602	.610	.617	.616	.615	.612	.602	.622
1.800	.601	.609	.615	.615	.614	.612	.602	.621
1.900	.601	.608	.614	.613	.612	.611	.602	.621
2.000	.601	.607	.613	.612	.612	.611	.602	.620
3.000	.601	.603	.606	.608	.610	.609	.601	.615

EXAMPLE. — With $h_1 = 4$ in. [$= 0.10$ met.], $a = 8$ in. [$= 0.20$ met.], $b = 1$ ft. 8 in. [$= 0.51$ met.], required the (actual) volume discharged per second. See Fig. 553.

From the foregoing table,

for $h_1 = 0.10^{\text{m.}}$, $b = 0.60^{\text{m.}}$ and $a = 0.20^{\text{m.}}$, we find $\mu_0 = .602$
" $h_1 = 0.10^{\text{m.}}$, $b = 0.20^{\text{m.}}$ " $a = 0.20^{\text{m.}}$, " $\mu_0 = .592$
diff. $= .010$

Hence, by interpolation,

for $h_1 = 0.10^{\text{m.}}$ $b = 0.51^{\text{m.}}$, and $a = 0.20^{\text{m.}}$, we have

$$\mu_0 = 0.602 - \tfrac{9}{40}\,[0.602 - 0.592] = 0.600.$$

Hence [ft., lb., sec.], remembering that μ_0 is an abstract number, from eq. (6),

$$Q = 0.600 \times \tfrac{8}{12} \times \tfrac{20}{12}\sqrt{2 \times 32.2\left(\tfrac{4}{12} + \tfrac{1}{2}\cdot\tfrac{8}{12}\right)} = 4.36$$

cub. ft. per second.

CASE II. *Incomplete Contraction.*—This name is given to the cases, like those shown in Fig. 554, where one or more sides of the orifice have an interior border flush with the sides or bottom of the (square-cornered) tank.

Not only is the general direction of the stream altered, but the discharge is *greater*, on account of the larger size of the contracted section, since contraction is prevented on those sides which have a border. It is assumed that the contraction which *does* occur (on the other edges) is *perfect;* i.e., the cross-section of the tank is large compared with the orifice. According to the experiments of Bidone and Weisbach with Poncelet's orifices (i.e., orifices in thin plate mentioned in the preceding table), the actual volume discharged per unit of time is

$$Q = \mu ab\sqrt{2g\left(h_1 + \frac{a}{2}\right)}\,. \quad (7)$$

(differing from eq. (6) only in the coefficient of efflux μ), in which the abstract number μ is found thus: Determine a coefficient of efflux μ_0 as if eq. (6) were to be used in Case I; i.e., as if contraction were complete and perfect; then write

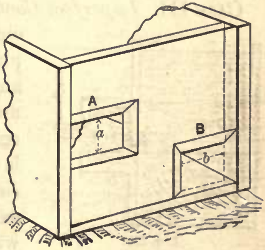

FIG. 554.

$$\mu = \mu_0[1 + 0.155n], \quad . \quad . \quad . \quad . \quad . \quad (7)'$$

where $n =$ the ratio of the length of periphery of the orifice with a border to the whole periphery.

E.g., if the lower sill, only, has a border,

$$n = b \div [2(a + b)];$$

while if the lower sill and both sides have a border,

$$n = (2a + b) \div [2(a + b)].$$

Example.—If $h_1 = 8$ ft. $(= 2.43^{m.})$, $b = 2$ ft. $(= 0.60^{m.})$, $a = 4$ in. $(= 0.10^{m.})$, and one side is even with the side of the tank, and the lower sill even with the bottom, required the volume discharged per second. (Sharp-edged orifice, in vertical plane, etc.)

Here for complete and perfect contraction we have, from Poncelet's tables (Case I), $\mu_0 = 0.608$. Now $n = \frac{1}{2}$; hence, from eq. (7)',

$$\mu = 0.608\,[1 + 0.155 \times \tfrac{1}{2}] = 0.6551;$$

hence, eq. (7),

$$Q = 0.655 \times 2 \times \tfrac{4}{12}\sqrt{2 \times 32.2(8 + \tfrac{1}{2} \cdot \tfrac{2}{12})}$$

$$= 10.23 \text{ cub. ft. per sec.}$$

Case III. *Imperfect Contraction.*—If there is a submerged channel of approach, symmetrically placed as regards the orifice, and of an area (cross-section), $= G$, not much larger than that, $= F$, of the orifice (see Fig. 555), the contraction is less than in Case I, and is called *imperfect contraction.* Upon his experiments with Poncelet's orifices, with imperfect contraction, Weisbach bases the following formula for the discharge (volume) per unit of time, viz.,

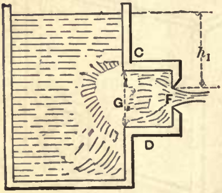

Fig. 555.

$$Q = \mu ab\sqrt{2g\left(h_1 + \frac{a}{2}\right)} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(see Fig. 553 for notation), with the understanding that the coefficient

$$\mu = \mu_0(1 + \beta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)'$$

where μ_0 is the coefficient obtained from the tables of Case I (as if the contraction were perfect and complete), and β an abstract number depending on the ratio $F : G = m$, as follows:

$$\beta = 0.0760\,[9^{m} - 1.00]. \quad . \quad . \quad . \quad . \quad . \quad (8)''$$

To shorten computation Weisbach gives the following table for β:

TABLE A.

m.	β.	m.	β.
.05	.009	.55	.178
.10	.019	.60	.208
.15	.030	.65	.241
.20	.042	.70	.278
.25	.056	.75	.319
.30	.071	.80	.365
.35	.088	.85	.416
.40	107	.90	.473
.45	.128	.95	.537
.50	.152	1.00	.608

EXAMPLE.—Let $h_1 = 4'\,9\tfrac{1}{2}''$ (= 1.46 met.), the dimensions of the orifice being—

width $= b = 8$ in. (= $0.20^{\text{m.}}$);

height $= a = 5$ in. (= $0.126^{\text{m.}}$);

while the channel of approach (*CD*, Fig. 555) is one foot square. From Case I, we have, for the given dimensions and head,

$$\mu_0 = 0.610;$$

$$\frac{F}{G} = \frac{40 \text{ sq. in.}}{144 \text{ sq. in.}} = 0.27.$$

We find [Table A]

$$\beta = 0.062;$$

and hence $\mu = \mu_0\,(1.062)$, from eq. (8)′. Therefore, from eq. (8), with ft., lb., and sec.,

$$Q = 0.610 \times 1.062 \times \tfrac{5}{12} \cdot \tfrac{8}{12} \sqrt{2 \times 32.2 \times 5}$$

$$= 3.22 \text{ cub. ft. per sec.}$$

CASE IV. *Head measured in Moving Water.*—See Fig. 556. If the head h_1, of the upper sill, cannot be measured to the level of still water, but must be taken to the surface of a channel of approach, where the velocity of approach is quite appreciable, not only is the contraction imperfect, but strictly we should use eq. (1) of § 501, in which the velocity of approach is considered. Let $F =$ area of orifice, and G that of the cross-section of the channel of approach; then the velocity of approach is $c = Q \div G$, and k (of above eq.) $= c^2 \div 2g = Q^2 \div 2gG^2$; but Q itself being unknown, a substitution of k in terms of Q in eq. (1), § 501, leads to an equation of high degree with respect to Q. Practically, therefore, it is better to write

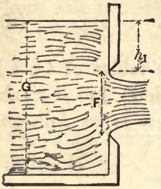

FIG. 556.

$$Q = \mu ab\sqrt{2g\left(h_1 + \frac{a}{2}\right)}, \quad . \quad . \quad . \quad . \quad (9)$$

and determine μ by experiment for different values of the ratio $F \div G$. Accordingly, Weisbach found, for Poncelet's orifices, that if μ_0 is the coefficient for complete and perfect contraction from Case I, we have

$$\mu = \mu_0(1 + \beta'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)'$$

β' being an abstract number, and being thus related to $F \div G$,

$$\beta' = 0.641\left(\frac{F}{G}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)''$$

h_1 was measured to the surface one metre back of the plane of the orifice, and $F : G$ did not exceed 0.50.

Weisbach gives the following table computed from eq. (9)″:

TABLE B.

$F \div G$.	β'.
0.05	.002
.10	.006
.15	.014
.20	.026
.25	.040
.30	.058
.35	.079
.40	.103
.45	.130
.50	.160

EXAMPLE.—A rectangular water-trough 4 ft. wide is dammed up with a vertical board in which is a rectangular orifice, as in Fig. 556, of width $b = 2$ ft. ($= 0.60$ met.), and height $a = 6$ in. ($= 0.15$ met.); and when the water-level behind the board has ceased rising (i.e., when the flow has become *steady*), we find that $h_1 = 2$ ft., and the depth behind in the trough to be 3 ft. Required Q.

Since $F : G = 1$ sq. ft. $\div$ 12 sq. ft. $= .0833$, we find (Table B) $\beta' = 0.005$; and μ_0 being $= 0.612$ from Poncelet's tables, Case I, we have finally, from eq. (9),

$$Q = 0.612\,(1.005)\,2 \times \tfrac{1}{2}\sqrt{2 \times 32.2 \times 2.25}$$
$$= 7.41 \text{ cub. ft. per second.}$$

504. Actual Discharge of Sharp-edged Overfalls (Overfall-weirs; or Rectangular Notches in a Thin Vertical Plate).

CASE I. *Complete and Perfect Contraction* (*the normal case*), Fig. 557; i.e., no edge is flush with the side or bottom of the reservoir, whose sectional area is very large compared with that, bh_2, of the notch. By depth, h_2, of the notch, we are to understand the *depth of the sill below the surface a few feet back of the notch where it is level*. In the plane of the notch the vertical thickness of the stream is only from $\frac{3}{4}$ to $\frac{9}{10}$ of h_2. Putting, therefore, the velocity of approach = zero, and hence $k = 0$, in eq. (3) of § 501, we have for the

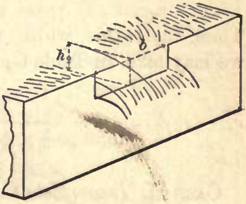

FIG. 557.

$$\textit{Actual } Q = \mu_0 \tfrac{2}{3} b h_2 \sqrt{2gh_2}, \quad . \quad . \quad . \quad . \quad (10)$$

(b = width of notch,) where μ_0 is a *coefficient of efflux* to be determined by experiment.

Experiments with overfalls do not agree as well as might be desired. Those of Poncelet and Lesbros gave the results in Table C.

TABLE C.

For $b = 0.20^{\text{m.}}$		For $b = 0.60^{\text{m.}}$	
metres.		metres.	
h_2	μ_0	h_2	μ_0
.01	.636	.06	.618
.02	.620	.08	.613
.03	.618	.10	.609
.04	.610	.12	.605
.06	.601	.15	.600
.08	.595	.20	.592
.10	.592	.30	.586
.15	.589	.40	.586
.20	.585	.50	.586
.22	.577	.60	.585
For approx. results $\mu_0 = .60$			

EXAMPLE 1.—With

$h_2 = 1$ ft. 4 in. ($= .405^{\text{m.}}$),

$b = 2$ ft. ($= 0.60^{\text{m.}}$),

we have, from Table C, $\mu_0 = .586$, and (ft., lb., sec.)

$$\therefore Q = .586 \times \tfrac{2}{3} \times 2 \times \tfrac{4}{3}\sqrt{2 \times 32.2 \times \tfrac{4}{3}}$$
$$= 9.54 \text{ cub. ft. per sec.}$$

EXAMPLE 2.—What width, b, must be given to a rectangular notch, for which $h_2 = 10$ in. ($= 0.25^{\text{m.}}$), that the discharge may be $Q = 6$ cub. feet per sec.?

Since b is unknown, we cannot use the table immediately, but take $\mu_0 = .600$ for a first approximation; whence, eq. (10), (ft., lb., sec.,)

$$b = \frac{6}{0.6 \times \frac{2}{3} \times \frac{10}{12} \sqrt{2 \times 32.2 \times \frac{10}{12}}} = 2.46 \text{ ft.}$$

Then, since this width does not much exceed 0.60 metre, we may take, in Table C, for $h_2 = 0.25$ met., $\mu_0 = .589$;

$$\therefore \quad b = \frac{6}{.589 \times \frac{2}{3} \times \frac{10}{12} \sqrt{2 \times 32.2 \times \frac{10}{12}}} = 2.50 \text{ ft.}$$

Case II. *Incomplete Contraction; i.e., both ends are flush with the sides of the tank, these being* ⊥ *to the plane of the notch.* According to Weisbach, we may write

$$Q = \tfrac{2}{3}\mu b h_2 \sqrt{2gh_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which $\mu = 1.041\mu_0$, μ_0 being obtained from Table C for the normal case, i.e., Case I. The section of channel of approach is large compared with that of the notch; if not, see Case IV.

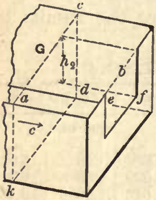

Fig. 558.

Case III. *Imperfect Contraction; i.e., the velocity of approach is appreciable;* the sectional area G of the channel of approach not being much larger than that, $F_1 = bh_2$ = area of notch. Fig. 558. b = width, and h_2 = depth of notch (see Case I). Here, instead of using a formula involving

$$k = c^2 \div 2g = [Q \div G]^2 \div 2g$$

(see eq. (3), § 501), it is more convenient to put

$$Q = \tfrac{2}{3}\mu b h_2 \sqrt{2gh_2}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

as before, with

$$\mu = \mu_0 (1 + \beta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)'$$

in which μ_0 is for the normal case [Case I]; and β, according to Weisbach's experiments, may be obtained from the empirical formula

$$\beta = 1.718\left(\frac{F}{G}\right)^4 . \quad . \quad . \quad . \quad . \quad . \qquad (12)''$$

[Table D is computed from (12)''.]

(The contraction is *complete* in this case; i.e., the ends are not flush with the sides of the tank.)

TABLE D.

$\frac{F}{G}$	β.
0.05	.000
.10	.000
.15	.001
.20	.003
.25	.007
.30	.014
.35	.026
.40	.044
.45	.070
.50	.107

EXAMPLE.—If the water in the channel of approach has a vertical transverse section of $G = 9$ sq. feet, while the notch is 2 feet wide (i.e., $b = 2'$) and 1 foot deep ($h_2 = 1'$) (to level of surface of water 3 or 4 ft. back of notch), we have, from Table C, with $b = .60$ met. and $h_2 = 0.30$ met.,

$$\mu_0 = 0.586;$$

while from Table D, with $F : G = 0.222$ (or $\frac{2}{9}$),

$$\beta = .005;$$

hence (ft.-lb.-sec. system of units), from eq. (12),

$$Q = \tfrac{2}{3} \times 0.586 \times 1.005 \times 2 \times 1 \times \sqrt{64.4 \times 1.0}$$

$$= 6.30 \text{ cub. ft. per second.}$$

CASE IV. Fig. 559. *Imperfect and incomplete contraction together;* both end-contractions being "suppressed" (by making the ends flush with the sides of the reservoir, these sides being vertical and ⊥ to the plane of the notch), and the channel of approach not being very deep, i.e., having a sectional area G but little larger than that, F, of notch. $F = bh_2$ as before.

Again we write

$$Q = \tfrac{2}{3}\mu b h_2 \sqrt{2gh_2}, \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

with μ computed from

$$\mu = \mu_0(1 + \beta), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)'$$

μ_0 being obtained from Table C; while

$$\beta = 0.041 + 0.3693 \left(\frac{F}{G}\right)^2, \quad . \quad . \quad . \quad . \quad (13)''$$

an empirical formula based by Weisbach on his own experiments.* To save computation, β may be found from Table E, founded on eq. (13)″.

TABLE E.

$\frac{F}{G}=$	.00	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
$\beta=$	.041	.042	.045	.049	.056	.064	.074	.086	.100	.116	.133

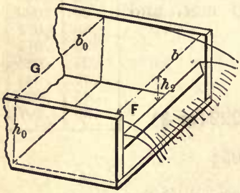

FIG. 559.

EXAMPLE.—Fig. 559. With

$$b = 2 \text{ ft. } (=0.60 \text{ met.})$$

and

$$h_2 = 1 \text{ ft. } (= 0.30 \text{ met.}),$$

we have, from Table C, $\mu_0 = 0.586$. But, the ends being flush with the sides of the reservoir or channel, and G being $= 6$ sq. ft. (see figure), which is not excessively large compared with $F = bh_2 = 2$ sq. ft., we have, from Table E, with $F : G = 0.333$,

$$\beta = .081;$$

and hence [eq. (13) and (13)′], μ_0 being .586 as in last example,

$$Q = \tfrac{2}{3} \times 0.586 \times (1 + .081) \times 2 \times 1 \times \sqrt{64.4 \times 1.0}$$

$$= 6.78 \text{ cub. ft. per sec.}$$

505. Francis' Formula for Overfalls (i.e., rectangular notches).—From extensive experiments at Lowell, Mass., in 1851, with rectangular overfall-weirs, Mr. J. B. Francis deduced the following formula for the volume, Q, of flow per second over such weirs 10 feet in width, and with h_2 varying from 0.6 to 1.6 feet (from sill of notch to level surface of water a few feet back):

---

* Weisbach's results in this case differ considerably from those of Bazin; see p. 688.

$$Q = \tfrac{2}{3} \times 0.622 h_2 (b - \tfrac{1}{10} n h_2) \sqrt{2gh_2}, \quad . \quad . \quad (14)$$

in which $b =$ width.

This provides for incomplete contraction, as well as for complete and perfect contraction, by making

$n = 2$ for perfect and complete contraction (Fig. 557);
$n = 1$ when one end only is flush with side of channel;
$n = 0$ when both ends are flush with sides of channel.

The contraction was considered complete and perfect when the channel of approach was made as wide as practicable, = 13.96 feet, the depth being about 5 feet.

Mr. Francis also experimented with submerged or "drowned" weirs in 1883; such a weir being one in which the sill is below the level of the tail-water (i.e., of receiving channel).

506. Fteley and Stearns's Experiments at Boston, Mass., in 1877 and 1880.—These may be found in the Transactions of the American Society of Civil Engineers, vol. XII, and gave rise to formulæ differing slightly from those of Mr. Francis in some particulars. In the case of *suppressed end-contractions*, like that in Fig. 559, they propose formulæ as follows:

When depth of notch is not large,

$$Q \text{ (in cub. ft. per sec.)} = 3.31\, b h_2^{\frac{3}{2}} + 0.007\, b \quad . \quad (15)$$

(*b and* h_2 *both in feet*),

"h_2, the depth on the weir, should be measured from the surface of the water above the curvature of the sheet."

"Air should have free access to the space under the sheet." The crest must be horizontal. The formula does not apply to depths on the weir less than 0.07 feet.

When the depth of notch is quite large, a correction must be made for velocity of approach, c, thus:

$$Q \text{ (in cub. ft. per sec.)} = 3.31\, b \left[h_2 + 1.5 \frac{c^2}{2g} \right]^{\frac{3}{2}} + 0.007\, b \quad (16)$$

(*b and* h_2 *both in feet*).

The channel should be of uniform rectangular section for about 20 ft. or more from the weir, to make this correction properly. If $G =$ the cross-section, in sq. ft., of the channel of approach, c is found approximately by dividing an approximate value of Q by G; and so on for closer results.

The weir may be of any length, b, from 5 to 19 feet.

506a. Recent Experiments on Overfall-weirs in France.—In the *Annales des Ponts et Chaussées* for October 1888 is an account of extensive and careful experiments conducted in 1886 and 1887 by M. Bazin on the flow over sharp-edged overfall-weirs with end-contractions suppressed; i.e., like that shown in Fig. 559. The widths of the weirs ranged from 0.50 to 2.00 metres, and the depths on the weirs (h_2) from 0.05 to 0.60 metre. With p indicating the height of the sill of the weir from the bottom of the channel of approach, M. Bazin, as a practical result of the experiments, recommends the following formula as giving a reasonably accurate value for the volume of discharge per unit of time:

$$Q = \tfrac{2}{3}\mu' \left[1 + 0.55\left(\frac{h_2}{p + h_2}\right)^2\right] bh_2 \sqrt{2gh_2}, \quad . \quad . \quad \mathbf{(17)}$$

where the coefficient μ' has a value

$$\mu' = 0.6075 + \frac{0.0148}{h_2\,(\text{in ft.})}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Eq. (17) is homogeneous, i.e., admits of any system of units.

Provision was made in these experiments for the free entrance of air under the sheet (a point of great importance), while the walls of the channel of approach were continued down-stream, beyond the plane of the weir, to prevent any lateral expansion of the sheet. The value of p ranged from 0.20 to 2.00 metres.

Herr Ritter von Wex in his "*Hydrodynamik*" (Leipsic, 1888) derives formulæ for weirs, in the establishing of which some rather peculiar views in the Mechanics of Fluids are advanced.

Formulæ and tables for discharge through orifices or over weirs of some forms not given here may be found in the works of Weisbach, Rankine, and Trautwine.

Mr. Hamilton Smith, a noted American hydraulic engineer, has recently published "Hydraulics," a valuable compilation and résumé of the most trustworthy experiments in all fields of hydraulics (New York, 1886: John Wiley & Sons).

507. Efflux through Short Cylindrical Tubes.—When efflux takes place through a short cylindrical tube, or "short pipe," at least $2\frac{1}{2}$ times as long as wide, inserted at right angles in the plane side of a large reservoir, the inner corners *not rounded* (see Fig. 560), the jet issues from the tube in parallel filaments and with a sectional area, F_m, equal to that, F, of interior of tube.

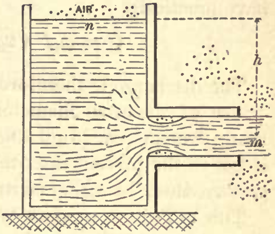

FIG. 560.

To attain this result, however, the tube must be full of water before the outer end is unstopped, and must not be oily; nor must the head, h, be greater than about 40 ft. for efflux into the air. Since at m the filaments are parallel and the pressure-head therefore equal to b (= 34 ft. of water, nearly), = that of surrounding medium, = head due to one atmosphere in this instance; an application of Bernoulli's Theorem [eq. (7), § 492] to positions m and n would give (precisely as in §§ 454 and 455)

$$v_m = \text{veloc. at } m = \sqrt{2gh}$$

as a theoretical result; but experiment shows that the actual value of v_m in this case is

$$v_m = \phi_0 \sqrt{2gh} = 0.815 \sqrt{2gh}, \quad . \quad . \quad . \quad . \quad (1)$$

0.815 being an average value for ϕ_0, the *coefficient of velocity*, for ordinary purposes. It increases slightly as the head decreases, and is evidently much less than the value 0.97 for an orifice in a thin plate, § 495, or for a rounded mouth-piece as in § 496.

But as the sectional area of the stream where the filaments are parallel, at m, where $v_m = 0.815\sqrt{2gh}$, is also equal to that, F, *of the tube*, the coefficient of efflux, μ_0, in the formula

$$Q = \mu_0 F\sqrt{2gh},$$

is equal to ϕ_0; i.e., there is no contraction, or the coefficient of contraction, C, in this case $= 1.00$.

Hence, for the volume of discharge per unit of time, we have practically

$$Q = \phi_0 F\sqrt{2gh} = 0.815\, F\sqrt{2gh}. \quad . \quad . \quad . \quad (2)$$

The discharge is therefore about $\frac{1}{8}$ greater than through an orifice of the same diameter in a thin plate under the same head [compare eq. (3), § 495]; for although at m the velocity is less in the present case, the *sectional area of the stream is greater*, there being no contraction.

This difference in velocity is due principally to the fact that the entrance of the tube has square edges, so that the stream contracts (at m', Fig. 561) to a section smaller than that of the tube, and then *re-expands* to the full section, F, of tube. The eddying and accompanying internal friction caused by this re-expansion (or "sudden enlargement" of the stream) is the principal *resistance* which diminishes the velocity. It is noticeable, also, in this case that the jet is not limpid and clear, as from thin plate, but troubled and only translucent (like ground-glass). The internal pressure in the stream at m' is found to be *less than one atmosphere*, i.e. less than that at m, as shown experimentally by the sucking in of air when a small aperture is made in the tube op-

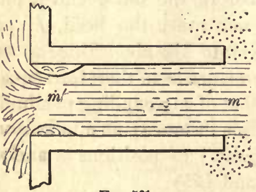

FIG. 561.

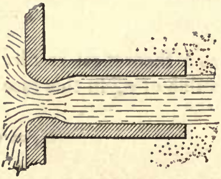

FIG. 562.

posite m'. If the tube itself were so formed internally as to fit this contracted vein, as in Fig. 562, the eddying would be diminished and the velocity at m increased, and hence the volume Q of efflux increased in the same proportion. (See § 509a.)

If the tube is less than $2\frac{1}{2}$ times as long as wide, or if the interior is *not wet by the water* (as when greasy), or if the head is over 40 or 50 ft. (about), the efflux takes place as if the tube were not there, Fig. 563, and we have

$$v_m = 0.97 \sqrt{2gh}, \text{ as in § 495.}$$

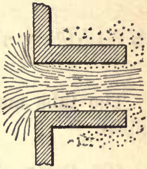

FIG. 563.

EXAMPLE.—The discharge through a short pipe 3 inches in diameter, like that in Fig. 560, is 30 cub. ft. per minute, under a head of 2′ 6″, reservoir large. Required the coefficient of efflux $\mu_0, = \phi_0$, in this case. For variety use the *inch-pound-minute* system of units, in which $g = 32.2 \times 12 \times 3600$ (see Note, § 51). μ_0, being an abstract number, will be the same numerically in any system of units.

From eq. (2),

$$\phi_0 = \mu_0 = \frac{Q}{F\sqrt{2gh}} = \frac{30 \times 1728}{\frac{\pi}{4} \times 3^2 \sqrt{2 \times 32.2 \times 12 \times 60^2 \times 30}}$$
$$= 0.803.$$

508. Inclined Short Tubes (Cylindrical).—Fig. 564. If the short tube is inclined at some angle $\alpha < 90°$ to the interior plane of the reservoir wall, the efflux is smaller than when the angle is 90°, as in § 507.

We still use the form of equation

$$Q = \mu F \sqrt{2gh} = \phi F \sqrt{2gh}; \quad . \quad (3)$$

but from Weisbach's experiments μ should be taken from the following table:

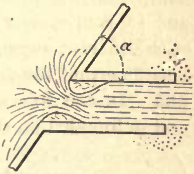

FIG. 564.

TABLE F, COEFFICIENT OF EFFLUX (INCLINED TUBE).

For $\alpha =$ 90°	80°	70°	60°	50°	40°	30°
take $\mu = \phi =$ .815	.799	.782	.764	.747	.731	.719

EXAMPLE.—With $h = 12$ ft., $d =$ diam. of tube $= 4$ ins., and $\alpha = 46°$, we have for the volume discharged per sec. (ft., lb., and sec.)

$$Q = [0.731 + \tfrac{6}{10}(.016)]\frac{\pi}{4}\left(\frac{1}{3}\right)^2 \sqrt{64.4 \times 12} = 1.79 \text{ cub.ft.per sec.}$$

The tube must be at least 3 times as long as wide, to be filled.

509. Conical Diverging, and Converging, Short Tubes.—With *conical convergent tubes*, as at A, Fig. 565, with inner edges not rounded, D'Aubuisson and Castel found by experiment values of the coefficient of velocity. ϕ, and of that of efflux, μ, [from which the coefficient of contraction, $C = \mu \div \phi$, may be computed,] for tubes 1.55 centimeters wide at the narrow end, and 4.0 centimeters long, under a head of $h = 3$ metres. and with different angles of convergence. By angle of convergence is meant the angle between the sides CE and DB. Fig. 565. In the following table will be found some values of μ and ϕ founded on these experiments, for use in the formulæ

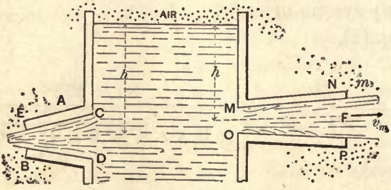

FIG. 565.

$$v_m = \phi \sqrt{2gh} \quad \text{and} \quad Q = \mu F \sqrt{2gh};$$

in which F denotes the area of the outlet orifice EB.

TABLE G (CONICAL CONVERGING TUBES).

Angle of convergence =	3° 10′	8°	10° 20′	13° 30′	19° 30′	30°	49°
μ =	.895	.930	.938	.946	.924	.895	.847
ϕ =	.894	.932	.951	.963	.970	.975	.984

Evidently μ is a maximum for $13\frac{1}{2}°$.

With a conically divergent tube as at MN, having the internal diameter $MO = .025$ metre, the internal diam. $NP = .032$ metre, and the angle between MN and $PO = 4° 50'$, Weisbach found that in the equation $Q = \mu F \sqrt{2gh}$ (where $F =$ area of *outlet* section NP) μ should be $= 0.553$; the great loss of velocity as compared with $\sqrt{2gh}$ being due to the eddying in the re-expansion from the contracted section at M (corners *not rounded*), as occurs also in Fig. 549. The jet was much troubled and pulsated violently.

When the angle of divergence is too great, or the head h too large, or if the tube is *not wet* by the water, efflux with the tube filled cannot be maintained, the flow then taking place as in Fig. 563.

Venturi and Eytelwein experimented with a conically divergent tube (called now "*Venturi's tube*"), with rounded entrance to conform to the shape of the contracted vein, as in Fig. 566, having a diameter of one inch at m' (narrowest part), where the sectional area $= F' = 0.7854$ sq. in., and of 1.80 inches at m (outlet), where area $= F$; the length being 8 ins., and the angle of convergence 5° 9′.

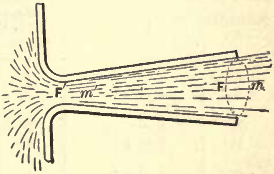

FIG. 566.

With $Q = \mu F \sqrt{2gh}$ they found $\mu = 0.483$.

Hence $2\frac{1}{2}$ times as much water was discharged as would have flowed out under the same head through an orifice in thin plate with area $= F' =$ the *smallest* section of the divergent tube, and 1.9 times as much as through a short pipe of section $= F'$. A similar calculation shows that the velocity at m' must have been $v_{m'} = 1.55 \sqrt{2gh}$, and hence that the pressure at m' was much less than one atmosphere.

Mr. J. B. Francis also experimented with Venturi's tube (see "Lowell Hydraulic Experiments"). See also p. 389 of vol. 6 of the Journal of Engineering Societies, for experiments with diverging short tubes discharging under water. The highest coefficient (μ) obtained by Mr. Francis was 0.782.

509a. New Forms of the Venturi Tube.—The statement made in § 507, in connection with Fig. 562, was based on purely theoretic grounds, but has recently (Dec. 1888) been completely verified by experiments* conducted in the hydraulic laboratory of the College of Civil Engineering at Cornell University. Three short tubes of circular section, each 3 in. in length and 1 in. in internal diameter at *both* ends, were experimented with, under heads of 2 ft. and 4 ft. Call them A, B, and C. A was an ordinary straight tube as in Fig. 561; the longitudinal section of B was like that in Fig. 562, the narrowest diameter being 0.80 in. [see § 495; $(0.8)^2 = 0.64$]; while C was somewhat like that in Fig. 566, being formed like B up to the narrowest part (diameter 0.80 in.), and then made conically divergent to the discharging end. The results of the experiments are given in the following table:

Name of Tube.	Head.	Number of Experiments.	Range of Values of μ.	Average Values of μ.
A	$h = 2$ ft.	4	From 0.804 to 0.823	0.814
A	$h = 4$ ft.	3	" 0.819 to 0.823	0.821
B	$h = 2$ ft.	5	" 0.875 to 0.886	0.882
B	$h = 4$ ft.	4	" 0.881 to 0.902	0.892
C	$h = 2$ ft.	5	" 0.890 to 0.919	0.901
C	$h = 4$ ft.	4	" 0.902 to 0.923	0.914

The fact that B discharges more than A is very noticeable, while the superiority of C to B, though evident, is not nearly so great as that of B to A, showing that in order to increase the discharge of an (originally) straight tube (by *encroaching* on the passage-way) it is of more importance to fill up with solid substance the space around the contracted vein than to make the transition from the narrow section to the discharging end very gradual.

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* See Journal of the Franklin Inst., for April, 1889.

510. "Fluid Friction."—By experimenting with the flow of water in glass pipes inserted in the side of a tank, Prof. Reynolds of England has found that the flow goes on in parallel filaments for only a few feet from the entrance of the tube, and that then the liquid particles begin to intermingle and cross each other's paths in the most intricate manner. To render this phenomenon visible, he injected a fine stream of colored liquid at the inlet of the pipe and observed its further motion, and found that the greater the velocity the nearer to the inlet was the point where the breaking up of the parallelism of flow began. The hypothesis of laminated flow is, nevertheless, the simplest theoretical basis for establishing practical formulæ, and the resistance offered by pipes to the flow of liquids in them will therefore be attributed to the friction of the edges of the laminæ against the inner surface of the pipe.

The amount of this resistance (often called *skin-friction*) for a given extent of rubbing surface is by experiment found—

1. To be *independent of the pressure* between the liquid and the solid;

2. To vary nearly with the *square of the relative velocity;*

3. To vary directly with the *amount of rubbing surface;*

4. To vary directly with the *heaviness* [γ, § 409] of the liquid.

Hence for a given velocity v, a given rubbing surface of area $= S$, and a liquid of heaviness γ, we may write

$$\text{Amount of friction (force)} = fS\gamma\frac{v^2}{2g}, \quad . \ . \ . \ . \ . \ . \ . \quad (1)$$

in which f is an abstract number called the *coefficient of fluid friction*, to be determined by experiment. For a given liquid, given character (roughness) of surface, and small range of velocities it is approximately constant. The object of introducing the $2g$ is not only because $\frac{v^2}{2g}$ is a familiar and useful function of v, but that $v^2 \div 2g$ is a *height*, or distance, and therefore the product of S (an area) by $v^2 \div 2g$ is a *volume*, and this volume multiplied by γ gives the *weight* of an ideal prism of the liquid; hence $S\frac{v^2}{2g}\gamma$ is a *force* and f must be *an abstract number* and therefore the same in all systems of units, in any given case or experiment.

In his experiments at Torquay, England, the late Mr. Froude found the following values for f, the liquid being salt water, while the rigid surfaces were the two sides of a thin straight wooden board $\frac{3}{16}$ of an inch thick and 19 inches high, coated or prepared in various ways, and drawn edgewise through the water at a constant velocity, the total resistance being measured by a dynamometer.

511. Mr. Froude's Results.—(Condensed.) [The velocity was the same = 10 ft. per sec. in each of the following cases. For other velocities the resistance was found to vary nearly as the square of the velocity, the index of the power varying from 1.8 to 2.16.]

TABLE H.

Character of Surface.	Value of f [from eq. (1), § 510].			
	When the board was			
	2 ft. long.	8 ft. long.	20 ft. long.	50 ft. long.
Varnish................ $f =$	0.0041	0.0032	0.0028	0.0025
Paraffine................ "	.0038	.0031	.0027	....
Tinfoil......................	.0030	0028	.0026	.0025
Calico.......................	.0087	.0063	.0053	.0047
Fine Sand..................	.0081	.0058	.0048	.0040
Medium Sand..............	.0090	.0062	.0053	.0049
Coarse Sand................	.0110	.0071	.0059	....

N.B. These numbers multiplied by 100 also give the mean frictional resistance in lbs. per sq. foot of area of surface in each case ($v = 10'$ per sec.), considering the heaviness of sea water, 64 lbs. per cubic foot, to cancel the $2g = 64.4$ ft. per sq. sec. of eq. (1) of the preceding paragraph.

For use in formulæ bearing on flow in pipes, f is best determined directly by experiments of that very nature, the results of which will be given as soon as the proper formulæ have been established.

512. Bernoulli's Theorem for Steady Flow, with Friction.—[The student will now re-read the first part of § 492, as far as eq. (1).] Considering free any lamina of fluid, Fig. 567, (according to the subdivision of the stream agreed upon in § 492 referred

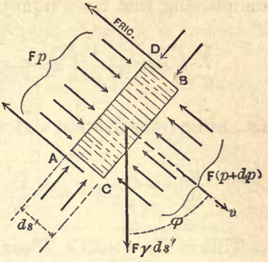

FIG. 567.

to,) the frictions on the edges are the only additional forces as compared with the system in Fig. 534. Let w denote the length of the *wetted perimeter* of the base of this lamina (in case of a pipe running full, as we here postulate, the wetted perimeter is of course the *whole perimeter*, but in the case of an open channel or canal, w is only a portion of the whole perimeter of the cross-section). Then, since the area of rubbing surface at the edge is $S = wds'$, the total friction for the lamina is [by eq. (1), § 510] $= fw\gamma(v^2 \div 2g)ds'$. Hence from $vdv =$ (tan. accel.) × ds, and from (tan. accel.) = [Σ(tang. compons. of acting forces)] ÷ (mass of lamina), we have

$$vdv = \frac{Fp - F(p+dp) + F\gamma ds' \cos\phi - fw\gamma \frac{v^2}{2g} ds'}{F\gamma ds' \div g} . ds ...(a)$$

As in § 492, so here, considering the simultaneous advance of all the laminæ lying between any two sections m and n during the small time dt, putting $ds' = ds$, and $ds' \cos\phi = -dz$ (see Fig. 568), we have, for any one lamina,

$$\frac{1}{g}vdv + \frac{1}{\gamma}dp + dz = -f\frac{w}{F} . \frac{v^2}{2g}ds. \quad . \quad . \quad . \quad (1)$$

Now conceive an infinite number of equations to be formed like eq. (1), one for each lamina between n and m, for *the same* dt, viz., a dt of such length that each lamina at the end of dt will occupy the same position, and acquire the same values of v, z, and p, that the lamina next in front had at the beginning of the dt (this is the characteristic of a *steady flow*). Adding up

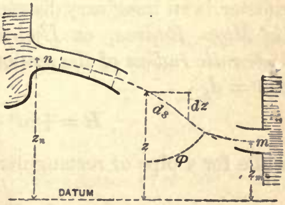

FIG. 568.

the corresponding terms of all these equations, we have (remembering that for a liquid γ is the same in all laminæ),

$$\frac{1}{g}\int_n^m v\,dv + \frac{1}{\gamma}\int_n^m dp + \int_n^m dz = -\frac{f}{2g} \cdot \int_n^m \frac{w}{F} v^2 ds\,; \quad . \quad (2)$$

i.e., after transposition and writing R for $F \div w$, for brevity,

$$\frac{v_m^2}{2g} + \frac{p_m}{\gamma} + z_m = \frac{v_n^2}{2g} + \frac{p_n}{\gamma} + z_n - \frac{f}{2g}\int_n^m \frac{v^2 ds}{R}. \quad . \quad (3)$$

This is *Bernoulli's Theorem for steady flow of a liquid in a pipe of slightly varying sectional area F, and internal perimeter w*, taking into account no resistances or friction, except the "skin-friction," or "fluid-friction," of the liquid and sides of the pipe.

Resistances due to the internal friction of eddying occasioned by sudden enlargements of the cross-section of the pipe, by elbows, sharp curves, valve-gates, etc., will be mentioned later. The negative term on the right in (3) is of course a *height* or *head* (one dimension of length), as all the other terms are such, and since it is the amount by which the sum of the three heads (viz., *velocity-head*, *pressure-head*, and *potential head*) at m, the down-stream position, *lacks* of being equal to the sum of the corresponding heads at n, the up-stream position or section, we may call it the "**Loss of Head**" due to skin-friction between n and m; also called *friction-head*, or *resistance-head*, or *height of resistance.*

The quantity $R = F \div w =$ sectional-area $\div$ wetted-perimeter, is an imaginary distance or length called the *Hydraulic Mean Radius*, or *Hydraulic Mean Depth*, or simply *hydraulic radius* of the section. For a circular pipe of diameter $= d$,

$$R = \tfrac{1}{4}\pi d^2 \div \pi d = \tfrac{1}{4}d\,;$$

while for a pipe of rectangular section,

$$R = \frac{ab}{2(a+b)}.$$

513. Problems involving Friction-heads; and Examples of Bernoulli's Theorem with Friction.

Problem I.—Let the portion of pipe between n and m be level, and of uniform circular section and diameter $= d$. The jet at m discharges into the air, and has the same sectional area, $F = \frac{1}{4}\pi d^2$, as the pipe; then the pressure-head at m is $\frac{p_m}{\gamma} = b =$ 34 feet (for water), and the velocity-head at m is = that at n, since $v_m = v_n$. The height of the water column in the open piezometer at n is noted, and $= y_n$ (so that the pressure-head at n is $\frac{p_n}{\gamma} = y_n + b$); while the length of pipe from n to m is $= l$.

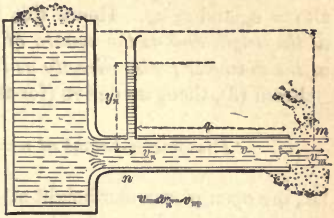

FIG. 569.

Knowing l, d, y_n, and having measured the volume Q, of flow, per unit of time, it is required to find the form of the friction-head and the value of f. From

$$F_m v_m = Q, \quad \text{or} \quad \frac{1}{4}\pi d^2 v_m = Q, \quad . \quad . \quad . \quad . \quad (1)$$

v_m becomes known. Also, v_m is known to be $= v_n$, and the velocity at each ds is $v = v_m$, since F (sectional area) is constant along the pipe, and $Fv = Q$. The hydraulic radius is

$$R = \frac{1}{4}d; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the same for all the ds's between n and m.

Substituting in eq. (3) of § 512, with the horizontal axis of the pipe as a datum for potential heads, we have

$$\frac{v_m^2}{2g} + b + 0 = \frac{v_n^2}{2g} + y_n + b + 0 - \frac{f}{\frac{1}{4}d} \cdot \frac{v_m^2}{2g}\int_n^m ds; \quad . \quad (3)$$

i.e., since $\int_n^m ds = l =$ length of pipe from n to m, *the friction-head for a pipe of length $= l$, and uniform circular section of diameter $= d$, reduces to the form*

$$\textit{Friction-head} = 4f\frac{l}{d}\cdot\frac{v^2}{2g}; \quad . \quad . \quad . \quad . \quad (4)$$

where $v =$ velocity of water *in the pipe*, being in this case also $= v_m$ and $= v_n$. Hence this friction-head *varies directly as the length and as the square of the velocity, and inversely as the diameter; also directly as the coefficient f.*

From (3), then, we derive (for this particular problem)

$$\textit{Piezometer-height at } n = y_n = 4f\frac{l}{d}\cdot\frac{v^2}{2g}; \quad . \quad . \quad (5)$$

i.e., the open piezometer-height at n is equal to the loss of head (all of which is friction-head here) sustained between n and the mouth of the pipe. (Pipe horizontal.)

Example.—Required the value of f, knowing that $d = 3$ in., y_n (by observation) $= 10.4$ ft., and $Q = 0.1960$ cub. ft. per sec., while $l = 400$ ft. (n to m). From eq. (1) we find, in ft.-lb.-sec. system, the velocity in the pipe to be

$$v = \frac{4Q}{\pi d^2} = \frac{4 \times 0.1960}{\pi \frac{1}{16}} = 4.0 \text{ ft. per sec.};$$

then, using eq. (5), we determine f to be

$$f = \frac{2gy_n d}{4lv^2} = \frac{2 \times 32.2 \times \frac{1}{4} \times 10.4}{4 \times 400 \times 4^2} = 0.0065.$$

Problem II. *Hydraulic Accumulator.*—Fig. 570. Let the area F_n of the piston on the left be quite large compared with that of the pipes and nozzle. The cylinder contains a friction

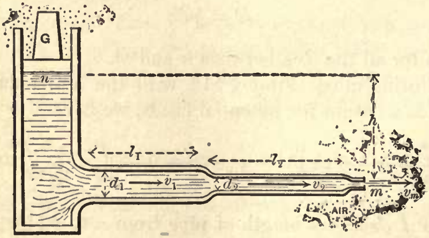

Fig. 570.

less weighted piston, producing (so long as its downward *slow* motion is uniform) a fluid pressure on its lower face of an intensity $p_n = [G + F_n p_a] \div F_n$ per unit area (p_a = one atmos.).

Hence the pressure-head at n is

$$\frac{p_n}{\gamma} = \frac{G}{F_n \gamma} + b, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where G = load on piston.

The jet has a section at $m = F_m$ = that of the small straight nozzle (no contraction). The junctions of the pipes with each other, and with the cylinder and nozzle, are all smoothly rounded; hence the only losses of head in steady flow between n and m are the friction-heads in the two long pipes, neglecting that in the short nozzle. These friction-heads will be of the form in eq. (4), and will involve the velocities v_1 and v_2 respectively in these pipes (*supposed* running full). v_1 and v_2 may be unknown at the outset, as here.

Knowing G and all dimensions and heights, we are required to find the velocity v_m of the jet, flowing into the air, and the volume of flow, Q, per unit of time, assuming f to be known and to be the same in both pipes (not strictly true).

Let the lengths and diameters be denoted as in Fig. 570, their sectional areas F_1 and F_2, the unknown velocities in them v_1 and v_2.

From the *equation of continuity* [eq. (3), § 490], we have

$$v_1 = \frac{F_m v_m}{F_1} \quad \text{and} \quad v_2 = \frac{F_m v_m}{F_2}. \quad . \quad . \quad . \quad (7)$$

To find v_m, we apply Bernoulli's Theorem (with friction), eq. (3), § 512, taking the down-stream position m in the jet close to the nozzle, and the up-stream position n just under the piston in the cylinder where the velocity v_n is practically nothing. Then with m as datum plane we have

$$\frac{v_m^2}{2g} + b + 0 = 0 + \frac{p_n}{\gamma} + h - 4f\frac{l_1}{d_1} \cdot \frac{v_1^2}{2g} - 4f\frac{l_2}{d_2} \cdot \frac{v_2^2}{2g}. \quad (8)$$

Apparently (8) contains three unknown quantities, v_m, v_1, and v_2; but from eqs. (7) v_1 and v_2 can be expressed in terms of v_m, whence [see also eq. (6)]

$$\frac{v_m^2}{2g}\left[1+4f\frac{l_1}{d_1}\left(\frac{F_m}{F_1}\right)^2+4f\frac{l_2}{d_2}\left(\frac{F_m}{F_2}\right)^2\right]=h+\frac{G}{F_n\gamma}; \quad . \quad (9)$$

or, finally,

$$v_m=\frac{\sqrt{2g\left(h+\frac{G}{F_n\gamma}\right)}}{\sqrt{1+4f\frac{l_1}{d_1}\left(\frac{F_m}{F_1}\right)^2+4f\frac{l_2}{d_2}\left(\frac{F_m}{F_2}\right)^2}}; \quad . \quad . \quad (10)$$

and hence we have also

$$Q=F_m v_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

EXAMPLE.—If we replace the force G of this problem by the thrust P exerted along the pump-piston of a steam fire-engine, we may treat the foregoing as a close approximation to the practical problem of such an apparatus, the pipes being consecutive straight lengths of hose, in which (for the probable values of v_1 and v_2) we may take $f=.0075$ (see "Fire-streams," by Geo. Ellis, Springfield, Mass.).* (Strictly, f varies somewhat with the velocity; see § 517.) Let $P=12000$ lbs., and the piston-area at $n=F_n=72$ sq. in. $=\frac{1}{2}$ sq. ft. Also, let $h=20$ ft., and the dimensions of the hose be as follows:

$d_1=3$ in., $d_2=2$ in., d_n (of nozzle) $=1$ in.;

$l_1=400$ ft., $l_2=500$ ft.

With the foot-pound-second system of units, we now have [eq. (10)]

$$v_m=\sqrt{\frac{2\times 32.2\left[20+\frac{12000}{\frac{1}{2}\times 62.5}\right]}{1+4\times .0075\left[\frac{400}{\frac{1}{4}}\left(\frac{1}{9}\right)^2+\frac{500}{\frac{1}{6}}\left(\frac{1}{4}\right)^2\right]}}$$

---

* See § 578.

$$= \sqrt{\frac{2 \times 32.2 \times 404}{1 + 0.59 + 5.62}};$$

i.e., $v_m = 60.0$ ft. per sec. If this jet were directed vertically upward it should theoretically attain a height $= \frac{v_m^2}{2g} =$ nearly 56 feet, but the resistance of the air would reduce this to about 40 or 45 ft.

We have further, from eq. (1),

$$Q = F_m v_m = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 \times 60.0 = 3.27 \text{ cub. ft. per sec.}$$

If there were no resistance in the hose we should have, from § 497*a*,

$$v_m = \sqrt{2g\left[\frac{P}{F_n\gamma} + h\right]} = \sqrt{2 \times 32.2 \times 404} = 161.3 \text{ ft. per sec.}$$

513a. Influence of Changes of Temperature.—Although Poiseuille and Hagen found that with glass tubes of very small diameter the flow of water was increased threefold by a rise of temperature from 0° to 45° Cent., it is unlikely that with common pipes the rate of flow is appreciably affected by the ordinary fluctuations of temperature; at any rate, experiments of sufficient precision are wanting, as regards such an influence. See Mr. Hamilton Smith's "Hydraulics," p. 16, where he says: "Changes by variation in T (temperature) will probably only be appreciable with small orifices, or with very low heads for orifices or weirs."

514. Loss of Head in Orifices and Short Pipes.—So long as the steady flow between two localities n and m takes place in a pipe having *no abrupt* enlargement or diminution of section, nor sharp curves, bends, or elbows, the loss of head may be ascribed solely to the surface action (or "skin-friction") between water and pipe; but the introduction of any of the above-mentioned features occasions eddying and internal disturbance, and friction (and consequent heat): thereby causing further deviations from Bernoulli's Theorem; i.e., additional *losses of head*, or *heights of resistance.*

From the analogy of the form of a friction-head in a long pipe [eq. (4), § 513], we may assnme that any of the above heights of resistance is proportional to the square of the velocity, and may therefore always be written in the form

$$\left\{\begin{array}{l}\textit{Loss of Head due to any}\\ \textit{cause except skin-friction}\end{array}\right\} = \zeta\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which v is the velocity of the water in the pipe at the section where the resistance occurs; or if, on account of an abrupt enlargement of the stream-section, there is a corresponding diminution of velocity, then v is *always to denote this diminished velocity* (i.e., in the down-stream section). This velocity v is often an unknown at the outset.

ζ, corresponding to the abstract factor $4f\frac{l}{d}$ in the height of resistance due to skin-friction [eq. (4), § 513], is an abstract number called the **Coefficient of Resistance**, to be determined experimentally; or computed theoretically, where possible. Roughly speaking, it is independent of the velocity, for a given fitting, casing, pipe-joint, elbow, bend, valve-gate at a definite opening, etc., etc.

515. Heights of Resistance (or Losses of Head) Occasioned by Short Cylindrical Tubes.—When dealing with short tubes discharging into the air, in § 507, deviations from Bernoulli's Theorem were made good by using a *coefficient of velocity* ϕ, dependent on experiment. This device answered every purpose for the simple circumstances of the case, as well as for simple orifices. But the great variety of possible designs of a compound pipe (with skin-friction, bends, sudden changes of cross-section, etc.) renders it almost impossible, in such a pipe, to provide for deviations from Bernoulli's Theorem by a single coefficient of velocity (velocity of jet, that is) for the pipe *as a whole*, since new experiments would be needed for each new design of pipe. Hence the great utility of the conception of "loss of head," one for each source of resistance.

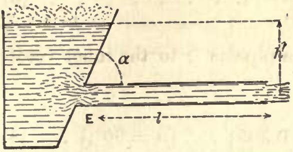

FIG. 571.

If a long pipe issues from the plane side of a reservoir and the corners of the junction are not rounded [see Fig. 571], we shall need an expression for the loss of head at the entrance, *E*, as well as that

$$\left[= 4f\frac{l}{d}\cdot\frac{v^2}{2g}\right]$$

due to the skin-friction in the pipe. But, whatever the velocity, v, in the pipe proves to be, influenced as it is both by the entrance loss of head and the skin-friction head (in applying Bernoulli's Theorem), the loss of head at *E*, viz., $\zeta_E\,\frac{v^2}{2g}$, will be just the same as if efflux took place through enough of the pipe at *E* to constitute a "short pipe," discharging into the air, under *some* head h (different from h' of Fig. 571) sufficient to produce the same velocity v. But in that case we should have

$$v = \phi\sqrt{2gh}, \quad \text{or} \quad \frac{v^2}{2g} = \phi^2 h. \quad . \quad . \quad . \quad . \quad (1)$$

(See §§ 507 and 508, ϕ being the "coefficient of velocity," and h the head, in the cases mentioned in those articles.)

We therefore apply Bernoulli's Theorem to the cases of those articles (see Figs. 560 and 564) in order to determine the loss of head due to the short pipe, and obtain (with m as datum level for potential heads)

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h - \zeta_E\frac{v^2}{2g}. \quad . \quad . \quad . \quad (2)$$

Now the v of eq. (2) is equal to the v_m of the figures referred to, and ζ_E is a *coefficient of resistance* for the short pipe, and we now desire its value. Substituting for

$$\frac{v^2}{2g}\left[= \frac{v_m^2}{2g}\right]$$

its value $\phi^2 h$ from eq. (1), we have

$$\zeta_E = \frac{1}{\phi^2} - 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence when $\alpha = 90°$ (i.e., the pipe is ⏋ to the inner reservoir surface), we derive

$$\zeta_E = \frac{1}{\phi_0^{\ 2}} - 1 = \frac{1}{(0.815)^2} - 1 = 0.505; \quad . \quad . [\alpha = 90°;] \quad . \quad . (4)$$

and similarly, for other values of α (taking ϕ from the table, § 508), we compute the following values of ζ_E (corners not rounded) for use in the expression for "loss of head," $\zeta_E \frac{v^2}{2g}$:

For $\alpha =$ 90°	80°	70°	60°	50°	40°	30°
$\zeta_E =$ .505	.565	.635	.713	.794	.870	.987

From eq. (4) we see that the loss of head at the entrance of the pipe, corners not rounded, with $\alpha = 90°$, is about one half (.505) of the height due to the velocity v in that part of the pipe (v being the same all along the pipe if cylindrical). The value of v itself, Fig. 571, depends on *all* the features of the design from reservoir to nozzle. See § 518.

If the corners at E are properly rounded, the entrance loss of head may practically be done away with; still. if v is quite small (as it may frequently be, from large losses of head farther down-stream), the saving thus secured, while helping to increase v slightly (and thus the saving itself), is insignificant.

516. General Form of Bernoulli's Theorem, considering all Losses of Head.

In view of preceding explanations and assumptions, we may write in a general and final form Bernoulli's Theorem for a steady flow from an up-stream position n to a down-stream position m, as follows:

$$\frac{v_m^{\ 2}}{2g} + \frac{p_m}{\gamma} + z_m = \frac{v_n^{\ 2}}{2g} + \frac{p_n}{\gamma} + z_n - \left\{ \begin{array}{c} \textit{all losses of head} \\ \textit{occurring between} \\ \textit{n and m} \end{array} \right\} . (B_f)$$

Each loss of head (or height of resistance) will be of the form $\zeta\frac{v^2}{2g}$ (except skin-friction head in long pipes, viz., $4f\frac{l}{d}\frac{v^2}{2g}$), the v in each case being the velocity, known or unknown, in that part of the pipe where the resistance occurs (and hence is not necessarily equal to v_m or v_n).

517. The Coefficient, *f*, for Friction of Water in Pipes.—See eq. (1), § 510. Experiments have been made by Weisbach, Eytelwein, Darcy, Bossut, Prony, Dubuat, Fanning, and others, to determine f in cylindrical pipes of various materials (tin, glass, zinc, lead, brass, cast and wrought iron) of diameters from ½ inch up to 36 inches. In general, the following deductions may be made from these experiments:

1st. f decreases when the velocity increases; e.g., in one case with the

same pipe f was $= .0070$ for $v = 2'$ per sec.,
while f was $= .0056$ for $v = 20'$ per sec.

2dly. f decreases slightly as the diameter increases (other things being equal);

e.g., in one experiment f was $= .0069$ in a 3-in. pipe,
while for the same velocity f was $= .0064$ in a 6-in. pipe.

3dly. The condition of the interior surface of the pipe affects the value of f, which is larger with increased roughness of pipe.

Thus, Darcy found, with a *foul* iron pipe with $d = 10$ in. and veloc. $\doteq 3.67$ ft. per sec.. the value .0113 for f; whereas Fanning (see p. 238 of his "Water-supply Engineering"), with a cement-lined pipe and velocity of 3.74 ft. per sec. and $d =$ 20 inches, obtained $f = .0052$.

Weisbach, finding the first relation very prominent, proposed the formula

$$f = 0.00359 + \frac{.00429}{\sqrt{v}\ \text{(in ft. per sec.)}}$$

when the velocities are great, while Darcy, taking into account both the 1st and 2d relations above, writes (see p. 585, Rankine's Applied Mechanics)

$$f = .0043\left[1 + \frac{1}{9 \times \text{diam. in ft.}}\right] +$$

$$+ \frac{.001}{v_{\text{ft. per sec.}}}\left[1 + \frac{1}{18 \times \text{diam. in ft.}}\right].$$

For practical purposes, Mr. J. T. Fanning has recommended, and arranged in an extensive table (pp. 242–246 of his book just mentioned), values of f for *clean iron pipe*, of diameters from ½ inch to 96 inches, and for velocities of 0.1 ft. to 20 ft. per second. Of this the table opposite is an abridgment, inserted with Mr. Fanning's permission, for use in solving numerical problems.

In obtaining f for *slightly tuberculated* and for *foul* pipes, the recommendations of Mr. Fanning seem to justify the following rules:

For *slightly tuberculated* pipes of diams. =	½ ft.	1 ft.	2 ft.	4 ft.
we should add	23%	34%	16%	13%
and for *foul* pipes of same size..........	72%	60%	38%	25%

of the f for clean pipes, to itself. For example, if $f = .007$ for a certain ½-ft. pipe when clean, with velocity = 0.64 ft. per sec., we have $f = .007 \times 1.72 = .01204$ when it is foul.

For first approximations a mean value of $f = .006$ may be employed, since in some problems sufficient data may not be known in advance to enable us to take f from the table.

EXAMPLE.—Fig. 572. In the steady pumping of crude petroleum weighing $\gamma = 55$ lbs. per cubic foot, through a six-inch pipe 30 miles long, to a station 700 ft. *higher* than the pump, it is found that the pressure in the pump cylinder at n, necessary to keep up a velocity of 4.4 ft. per sec. in the pipe, is 1000 lbs. per sq. inch. Required the coefficient f in the pipe. As all losses except the friction-head in the pipe are insignificant, the latter only will be considered. The velocity-head at n may

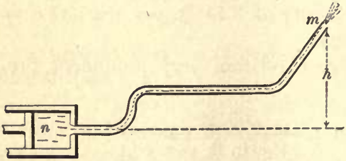

FIG. 572.

TABLE OF THE COEFFICIENT, *f*, FOR FRICTION OF WATER IN *CLEAN* IRON PIPES.

[Abridged from Fanning.]

Vel. in ft. per sec.	diam. = ½ in. =.0417ft.	diam. = 1 in. =.0834ft.	diam. = 2 in. =.1667ft.	diam. = 3 in. = .25 ft.	diam. = 4 in. = .333 ft.	diam. = 6 in. = .50 ft.	diam. = 8 in. = .667 ft.	diam. = 10 in. = .833 ft.	diam. = 12 in. = 1.00ft.	diam. = 16 in. =1.333ft.	diam. = 20 in. =1.667ft.	diam. = 30 in. = 2.50 ft.	diam. = 40 in. =3.333ft.	diam. = 60 in. = 5. ft.
0.1	.0150	.0119	.00870	.00800	.00763	.00730	.00704	.00684	.00669	.00623				
0.3	.0137	.0113	850	784	750	720	693	673	657	614	.00578			
0.6	.0124	.0104	822	767	732	702	677	659	642	603	567	.00504	.00434	.00357
1.0	.0110	.00950	790	743	712	684	659	643	624	588	555	492	428	353
1.5	.00959	.00868	.00757	.00720	.00693	.00662	.00640	.00625	.00607	.00572	.00542	.00482	.00421	.00349
2.0	.00862	810	731	700	678	648	624	609	593	559	529	470	416	346
2.5	795	768	710	683	662	634	611	596	581	548	518	460	410	342
3.0	.00753	.00734	.00692	.00670	.00650	.00623	.00600	.00584	.00570	.00538	.00509	.00452	.00407	00339
4.0	722	702	671	651	631	607	586	568	553	524	498	441	400	333
6.0	689	670	640	622	605	582	562	548	534	507	482	430	391	324
8.0	663	646	618	600	587	562	544	532	520	491	470	422	384	320
12.0	630	614	590	582	560	540	522	512	500	478	457	412	377	.00313
16.0	.00618	.00600	.00581	.00570	.00552	.00530	.00513	.00502	.00491	.00470	.00450	.00406	.00370	
20.0	615	598	579	566	549	525	508	498	485					

be put $= 0$; the jet at m being of the same size as the pipe, the velocity in the pipe is $= v_m$, and therefore $v_m = 4.4$ ft. per sec. Notice that m, the *down-stream* section, is at a *higher* level than n.

From Bernoulli's Theorem, § 516, we have, with n as a datum level,

$$\frac{v_m^2}{2g} + b + h = 0 + \frac{p_n}{\gamma} + 0 - 4f\frac{l}{d}\frac{v^2}{2g}. \quad . \quad . \quad (1)$$

Using the ft., lb., and sec., we have

$$h = 700 \text{ ft.}, \quad v_m^2 \div 2g = 0.30 \text{ ft.},$$

while

$$b = \frac{14.7 \times 144}{55} = 38.47 \text{ ft., and } \frac{p_n}{\gamma} = \frac{1000 \times 144}{55} = 2618 \text{ ft.}$$

Hence, in eq. (1),

$$0.30 + 38.5 + 700 = 2618 - 4f.\frac{30 \times 5280}{\frac{1}{2}} . \frac{(4.4)^2}{64.4}.$$

Solving for f, we have $f = .00485$ (whereas for water, with $v = 4.4$ ft. per sec. and $d = \frac{1}{2}$ ft., the table, p. 709, gives $f = .00601$.

If the γ of the petroleum had been 50 lbs. per cubic foot, instead of 55, we would have obtained $\frac{p_n}{\gamma} = 2880$ feet and $f = .0056$.

518. Flow through a Long Straight Cylindrical Pipe, including both friction-head and entrance loss of head (corners not rounded); reservoir large. Fig. 573. The jet issues directly from the end of the pipe, in parallel filaments, into the air, and therefore has same section as pipe; hence, also, v_m of the jet $= v$ in the pipe (which is assumed to be running full), and is

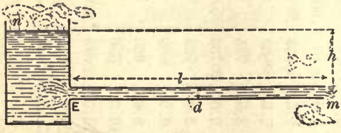

FIG. 573.

therefore the velocity to be used in the loss of head $\zeta_E \frac{v^2}{2g}$ at the entrance E (§ 515).

Taking m and n as in figure and applying Bernoulli's Theorem (§ 474), with m as datum level for the potential heads z_m and z_n, we have

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h - \zeta_E \frac{v^2}{2g} - 4f \frac{l}{d} \frac{v^2}{2g}. \quad . \quad (1)$$

Three different problems may now be solved:

First, required the head h to keep up a flow of given volume $= Q$ per unit of time in a pipe of given length l and diameter $= d$.

From the equation of continuity we have

$$Q = F_m v_m = \tfrac{1}{4}\pi d^2 v_m;$$

$$\therefore \textit{ veloc. of jet, which} = \textit{veloc. in pipe,} = v_m = \frac{4Q}{\pi d^2}. \quad . \quad . \quad (2)$$

Having found $v_m = v$, from (2), we obtain from (1) the required h, thus:

$$h = \frac{v^2}{2g}\left[1 + \zeta_E + 4f\frac{l}{d}\right]. \quad . \quad . \quad . \quad . \quad (3)$$

Now $\zeta_E = 0.505$ if $\alpha = 90°$ (see § 515), while f may be taken from the table, § 517, for the given diameter and computed velocity [$v_m = v$, found in (2)], if the pipe is *clean;* if not clean, see end of § 517, for *slightly tuberculated* and for *foul* pipes.

Secondly. Given the head h, and the length l and diameter d of pipe, required the velocity in the pipe, viz., $v, = v_m$, that of jet; also the volume delivered per unit of time, Q. Solving eq. (1) for v_m, we have

$$v_m = \sqrt{\frac{1}{1 + \zeta_E + 4f\frac{l}{d}}}\sqrt{2gh}; \quad . \quad . \quad . \quad (4)$$

whence Q becomes known, since

$$Q = \tfrac{1}{4}\pi d^2 v_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

[NOTE.—The first radical in (4) might for brevity be called a *coefficient of velocity*, ϕ, for this case. Since the jet has the same diameter as the pipe, this radical may also be called a *coefficient of efflux.*]

Since in (4) f depends on the unknown v as well as on the known d, we must first put $f = .006$ for a first approximation for v_m; then take a corresponding value for f and substitute again; and so on.

Thirdly, knowing the length of pipe and the head h, we wish to find the proper diameter d for the pipe to deliver a given volume Q of water per unit of time. Now

$$v, = v_m, = \frac{Q}{\frac{1}{4}\pi d^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which substituted in (1) gives

$$2gh = \left(\frac{4Q}{\pi}\right)^2 \frac{1}{d^4}\left[1 + \zeta_E + 4f\frac{l}{d}\right] = \left(\frac{4Q}{\pi}\right)^2 \left[\frac{1+\zeta_E}{d^4} + \frac{4fl}{d^5}\right];$$

that is,

$$2ghd^5 = \left(\frac{4Q}{\pi}\right)^2 [(1 + \zeta_E)d + 4fl];$$

$$\therefore\ d = \sqrt[5]{\frac{(1+\zeta_E)d + 4fl}{2gh} \cdot \left(\frac{4Q}{\pi}\right)^2} . \quad . \quad . \quad . \quad (7)$$

As the radical contains d, we first assume a value for d, with $f = .006$, and substitute in (7). With the approximate value of d thus obtained, we substitute again with a new value for f based on an approximate v from eq. (6) (with $d =$ its first approximation), and thus a still closer value for d is derived; and so on. (Trautwine's Pocket-book contains a table of fifth roots and powers.) If l is quite large, we may put $d = 0$ for a first approximation. In connection with these examples, see last figure.

EXAMPLE 1.—What head h is necessary to deliver 120 cub. ft. of water per minute through a clean straight iron pipe 140 ft. long and 6 in. in diameter?

From eq. (2), with ft., lb., and sec., we have

$$v = v_m = [4 \times \tfrac{120}{60}] \div \pi(\tfrac{1}{2})^2 = 10.18 \text{ ft. per sec.}$$

Now for $v = 10$ ft. per sec. and $d = \frac{1}{2}$ ft., we find (in table, § 517) $f = .00549$; and hence, from eq. (3),

$$h = \frac{(10.18)^2}{2 \times 32.2}\left[1 + .505 + \frac{4 \times .00549 \times 140}{\frac{1}{2}}\right] = 12.23 \text{ ft.},$$

of which total head, as we may call it, 1.60 ft. is used in producing the velocity 10.18 ft. per sec. (i.e., $v_m^2 \div 2g = 1.60$ ft.), while 0.808 ft. $\left(= \zeta_E \frac{v_m^2}{2g}\right)$ is lost at the entrance E (with $\alpha =$ 90°), and 9.82 ft. (friction-head) is lost in skin-friction.

EXAMPLE 2.—[Data from Weisbach.] Required the delivery, Q, through a straight clean iron pipe 48 ft. long and 2 in. in diameter, with 5 ft. head ($= h$). $v, = v_m$, being unknown, we first take $f = .006$ and obtain [eq. (4)]

$$v_m = \sqrt{\frac{1}{1 + .505 + \frac{4 \times .006 \times 48}{\frac{1}{6}}}}\sqrt{2 \times 32.2 \times 5}$$

$$= 6.18 \text{ ft. per sec.}$$

From the table, § 517, for $v = 6.2$ ft. per sec. and $d = 2$ in., $f = .00638$, whence

$$v_m = \sqrt{\frac{1}{1 + .505 + \frac{4 \times .00638 \times 48}{\frac{1}{6}}}}\sqrt{2 \times 32.2 \times 5}$$

$$= 6.04 \text{ ft. per sec.},$$

which is sufficiently close. Then, for the volume per second,

$$Q = \frac{\pi}{4} d^2 v_m = \tfrac{1}{4}\pi(\tfrac{1}{6})^2 6.04 = 0.1307 \text{ cub. ft. per sec.}$$

[Weisbach's results in this example are

$$v_m = 6.52 \text{ ft. per sec.}$$

and $$Q = 0.1420 \text{ cub. ft. per sec.,}$$

but his values for f are slightly different.]

EXAMPLE 3.—[Data from Weisbach.] What must be the diameter of a straight clean iron pipe 100 ft. in length, which is to deliver $Q = \frac{1}{2}$ of a cubic foot of water per second under 5 ft. head ($= h$)?

With $f = .006$ (approximately), we have from eq. (7), putting $d = 0$ under the radical for a first trial (ft., lb., sec.),

$$d = \sqrt[5]{\frac{4 \times .006 \times 100}{2 \times 32.2 \times 5} \cdot \left(\frac{\frac{4}{2}}{\pi}\right)^2} = \text{about } 0.30 \text{ ft.};$$

whence $$v = \frac{4Q}{\pi d^2} = 7 \text{ ft. per sec.}$$

For $d = 3.6$ in. and $v = 7$ ft. per sec., we find $f = .00601$; whence, again,

$$d = \sqrt[5]{\frac{1.505 \times .30 + 4 \times .00601 \times 100}{2 \times 32.2 \times 5} \cdot \left(\frac{4 \times \frac{1}{2}}{\pi}\right)^2} = 0.324 \text{ ft.};$$

and the corresponding $v = 6.06$ ft.

For this d and v we find $f = .00609$, whence, finally,

$$d = \sqrt[5]{\frac{1.505 \times .30 + 4 \times .00609 \times 100}{2 \times 32.2 \times 5} \left(\frac{2}{\pi}\right)^2} = 0.326 \text{ ft.}$$

[Weisbach's result is $d = .318$ ft.]

519. Chézy's Formula.—If, in the problem of the preceding paragraph, the pipe is *so long*, and therefore $l : d$ *so great*, that $4fl \div d$ in eq. (3) is very large compared with $1 + \zeta_E$, we may neglect the latter term without appreciable error; whence eq. (3) reduces to

$$h = 4f\frac{l}{d} \cdot \frac{v_m^2}{2g} \quad . \quad . \quad (\textit{pipe very long; } \text{Fig. 573}), \quad . \quad . \quad (8)$$

which is known as *Chézy's Formula.* For example, if $l = 1000$ ft. and $d = 2$ in. $= \frac{1}{6}$ ft., and f approx. $= .006$, we have $4f\frac{l}{d} = 144$, while $1 + \zeta_E$ for square corners $= 1.505$ only.

If in (8) we substitute

$$v_m = \frac{Q}{F_m} = Q \div \tfrac{1}{4}\pi d^2,$$

(8) reduces to

$$h = \frac{64}{\pi^2} \cdot f\frac{l}{d^5} \cdot \frac{Q^2}{2g} \quad . \; . \; . \quad (\textit{very long pipe}); \quad . \; . \; . \; . \; . \quad (9)$$

so that for a very long pipe, considering f as approximately constant, we may say that to deliver a volume $= Q$ per unit of time through a pipe of *given length* $= l$, the necessary head, h, is *inversely proportional to the fifth power of the diameter.*

And again, solving (9) for Q, we find that the volume conveyed per unit of time is directly proportional to the *fifth power of the square root of the diameter;* directly proportional to the *square root of the head;* and *inversely proportional to the square root of the length.* (Not true for short pipe; see above example.)

If we conceive of the insertion of a great number of piezometers along the long straight pipe, of uniform section, now under consideration, the summits of the respective water columns maintained in them will lie in a straight line joining the discharging (into the air) end of the pipe with a point in the reservoir surface vertically over the inlet extremity (practically so), and the "slope" of this line (called the *Hydraulic Grade Line* or *Gradient*), i.e., the tangent (or sine; the angle is so small, generally) of the angle which it makes with the horizontal is $= \frac{h}{l}$, and may be denoted by s. Putting also $\frac{1}{4}d = R =$ the hydraulic radius of the section of the pipe, and $v_m = v =$ velocity in pipe, we may transform eq. (8) into

$$v = \sqrt{\frac{2g}{f}}\,(Rs)^{\frac{1}{2}}; \quad \text{or,} \quad v = A(Rs)^{\frac{1}{2}}, \quad . \; . \; . \; (10)$$

which is the form by which Mr. Hamilton Smith (see § 506) interprets all the experiments quoted by him on long pipes. As to notation, however, he uses n for A, and r for R. With the foot and second as units, the quantity A (*not an abstract number*) varies approximately between 60 and 140. For a given A we easily find the corresponding f from the relation $f = \frac{2g}{A^2}$. *If the pipe discharges under water*, $h =$ the difference of elevation of the two reservoirs. If the pipe is not horizontal, the use of the length of its horizontal projection instead of its actual length in the relation $s = \frac{h}{l}$ occasions an error, but it is in most cases insignificant.

Similarly, if a steady flow is going on in a long pipe of uniform section, at the extremities of *any portion* of which we have measured the piezometer heights (or computed them from the readings of steam or pressure gauges), we may apply eq. (9), putting for h the difference of level of the piezometer summits, and for l the length of the pipe between them.

520. Coefficient f in Fire-engine Hose.*—Mr. Geo. A. Ellis, C.E., in his little book on "Fire-streams," describing experiments made in Springfield, Mass., gives a graphic comparison (p. 45 of his book) of the friction-heads occurring in *rubber hose*, in *leather hose*, and in *clean iron pipe*, each of 2½ in. diameter, with various velocities; on which the following statements may be based: That for the given size of hose and pipe ($d =$ 2½ in.) the coefficient f for the *leather* and *rubber* hose respectively may be obtained approximately by adding to f for clean iron pipe (and a given velocity) the per cent of itself shown in the accompanying table.

Velocity ft. per sec.	Rubber hose 2½ in. diam.	Leather hose 2½ in. diam.
3.0	50%	300%
6.5	20	80
10	16	43
13	12.5	32
16	12	30

EXAMPLE.—For a clean iron pipe 2½ in. diam., for a velocity = 10 ft. per sec., we have, from § 517, $f =$ .00593. Hence for a leather hose of the same diameter, we have, for $v =$ 10 ft. per sec.,

$$f = .00593 + .43 \times .00593 = .00848.$$

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* For the most recent and exhaustive experiments in this direction see Mr. J. R. Freeman's "Hydraulics of Fire-streams" in the Transac. Am. Soc. C. E. for 1889. (Also p. 832 of this work.)

521. Bernoulli's Theorem as an Expression of the Conservation of Energy for the Liquid Particles.—In any kind of flow *without friction, steady or not,* in *rigid immovable vessels*, the aggregate potential and kinetic energy of the whole mass of liquid concerned is necessarily a constant quantity (see §§ 148 and 149), but *individual particles* (as the particles in the sinking free surface of water in a vessel which is rapidly being emptied) may be continually losing potential energy, i.e., reaching lower and lower levels, without any compensating increase of kinetic energy or of any other kind; but in a *steady flow without friction in rigid motionless vessels*, we may state that the stock of energy of a given particle, or small collection of particles, is *constant* during the flow, provided we recognize a third kind of energy which may be called **Pressure-energy**, or capacity for doing work by virtue of internal fluid pressure; as may be thus explained:

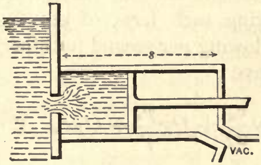

FIG. 574.

In Fig. 574 let water, with a very slow motion and under a pressure p (due to the reservoir-head + atmosphere-head behind it), be admitted behind a piston the space beyond which is *vacuous*. Let $s =$ length of stroke, and $F =$ the area of piston. At the end of the stroke, by motion of proper valves, communication with the reservoir is cut off on the left of the piston and opened on the right, while the water in the cylinder now on the left of the piston is put in communication with the vacuous exhaust-chamber. As a consequence the internal pressure of this water falls to zero (height of cylinder small), and on the return stroke is simply conveyed out of the cylinder, neither helping nor hindering the motion. That is, in doing the work of one stroke, viz.,

$$W = \text{force} \times \text{distance} = Fp \times s = Fps,$$

a volume of water $V = Fs$, weighing $Fs\gamma$ (lbs. or other unit), has been used, and, in passing through the motor, has experienced no appreciable change in velocity (motion slow), and therefore no change in kinetic energy, nor any change of level, and hence no change in potential energy, *but it has given up all its pressure.* (See § 409 for γ.)

Now W, the work obtained by the consumption of a weight $= G = V\gamma$ of water, may be written

$$W = Fps = Fsp = Vp = V\gamma \frac{p}{\gamma} = G\frac{p}{\gamma}. \quad . \quad . \quad (1)$$

Hence a weight of water $= G$ *is capable of doing the work* $G \times \frac{p}{\gamma} = G \times$ *head due to pressure* p, i.e., $= G \times$ *pressure-head, in giving up all its pressure* p; or otherwise, while still having a pressure p, a weight G of water possesses an *amount of energy*, which we may call *pressure-energy*, of an amount $= G \cdot \frac{p}{\gamma}$, where $\gamma =$ the heaviness (§ 7) of water, and $\frac{p}{\gamma} =$ a height, or head, measuring the pressure p; i.e., it equals the pressure-head.

We may now state Bernoulli's Theorem without friction in a new form, as follows: Multiplying each term of eq. (7), § 451, by $Q\gamma$, the weight of water flowing per second (or other time-unit) in the steady flow, we have

$$Q\gamma\frac{v_m^2}{2g} + Q\gamma\frac{p_m}{\gamma} + Q\gamma z_m = Q\gamma\frac{v_n^2}{2g} + Q\gamma\frac{p_n}{\gamma} + Q\gamma z_n. \quad (2)$$

But $Q\gamma\frac{v_m^2}{2g} = \frac{1}{2}\frac{Q\gamma}{g}v_m^2 = \frac{1}{2} \times$ *mass* flowing per time-unit × square of the velocity = the *kinetic energy* inherent in the volume Q of water on passing the section m, due to the velocity at m. Also, $Q\gamma\frac{p_m}{\gamma} =$ the *pressure-energy* of the volume Q at m, due to the pressure at m; while $Q\gamma z_m =$ the *potential energy* of the volume Q at m due to its height z_m above the arbitrary datum plane. Corresponding statements may be made for the terms on the right-hand side of (2) referring to the other section, n, of the pipe. Hence (2) may be thus read: *The aggregate amount of energy* (of the three kinds mentioned) *resident in the particles of liquid when passing section m is* *equal to that when passing any other section, as* n*; in steady flow without friction in rigid motionless vessels;* that is, the store of energy is constant.

522. Bernoulli's Theorem with Friction, from the Standpoint of Energy.—Multiply each term in the equation of § 516 by $Q\gamma$, as before, and denote a loss of head or height of resistance due to any cause by h_r, and we have

$$Q\gamma\frac{v_m^2}{2g}+Q\gamma\frac{p_m}{\gamma}+Q\gamma z_m$$

$$=Q\gamma\frac{v_n^2}{2g}+Q\gamma\frac{p_n}{\gamma}+Q\gamma z_n-\Sigma_n^m Q\gamma h_r \quad . \quad . \quad (3)$$

Each term $Q\gamma h_r$ (e.g., $Q\gamma\, 4f\,\frac{l}{d}\,\frac{v^2}{2g}$ due to skin-friction in a long pipe, and $Q\gamma\, \zeta_E\frac{v^2}{2g}$ due to loss of head at the reservoir entrance of a pipe) represents a *loss of energy*, occurring between any position n and any other position m down-stream from n, but is really still in existence in the form of heat generated by the friction of the liquid particles against each other or the sides of the pipes.

As illustrative of several points in this connection, consider two short lengths of pipe in Fig. 575, A and B, one offering a gradual, the other a sudden, enlargement of section, but *otherwise identical in dimensions.* We suppose them to occupy places in separate lines of pipe in each of which a steady flow with full cross-sections is proceeding, and so regulated that the velocity and internal pressure at n, in A, are equal respectively to those at n in B. Hence, if *vacuum* piezometers be inserted at n, the

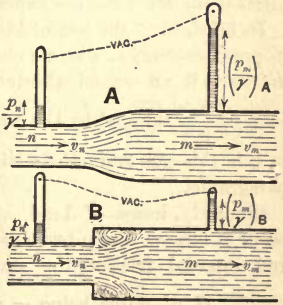

FIG. 575.

smaller section, the water columns maintained in them by the internal pressure will be of the same height, $\frac{p_n}{\gamma}$, for both A and B. Since at m, the larger section, the sectional area is the same for both A and B, and since F_n in $A = F_n$ in B, so that $Q_A = Q_B$, hence v_m in $A = v_m$ in B and is less than v_n.

Now in B a loss of head occurs (and hence a loss of energy) between n and m, but *none* in A (except slight friction-head); hence in A we should find as much energy present at m as at n, only differently distributed among the three kinds, while at m in B the aggregate energy is less than that at n in B.

As regards kinetic energy, there has been a loss between n and m in both A and B (and equal losses), for v_m is less than v_n. As to potential energy, there is no change between n and m either in A or B, since n and m are on a level. Hence if the loss of kinetic energy in B is not compensated for by an equal gain of pressure-energy (as it *is* in A), the pressure-head $\left(\frac{p_m}{\gamma}\right)_B$ at m in B should be less than that $\left(\frac{p_m}{\gamma}\right)_A$ at m in A. Experiment shows this to be true, the loss of head being due to the internal friction in the eddy occasioned by the sudden enlargement; the water column at m in B is found to be of a less height than that at m in A, whereas at n they are equal. (See p. 467 of article "Hydromechanics" in the *Ency. Britannica* for Mr. Froude's experiments.)

In brief, in A the loss of kinetic energy has been made up in pressure-energy, with no change of potential energy, but in B there is an actual absolute loss of energy $= Q\gamma h_r$, or $= Q\gamma\zeta\frac{v_m^2}{2g}$, suffered by the weight $Q\gamma$ of liquid. The value of ζ in this case and others will be considered in subsequent paragraphs.

Similarly, losses of head, and therefore losses of energy, occur at elbows, sharp bends, and obstructions, causing eddies and internal friction, the amount of each loss for a given weight, G, of water being $= Gh_r = G\zeta\frac{v^2}{2g}$; $h_r = \zeta\frac{v^2}{2g}$ being the loss of head occasioned by the obstruction (§ 474). It is therefore very important in transmitting water through pipes for purposes of *power* to use all possible means of preventing disturbance and eddying among the liquid particles. E.g., sharp corners, turns, elbows, abrupt changes of section, should be avoided in the design of the supply-pipe. The amount of the losses of head, or heights of resistance, due to these various causes will now be considered (except skin-friction, already treated). Each such loss of head will be written in the form $\zeta \frac{v^2}{2g}$, and we are principally concerned with the value of the abstract number ζ, or *coefficient of resistance*, in each case. The velocity v is the velocity, known or unknown, *where the resistance occurs;* or if the section of pipe changes at this place, then $v =$ velocity on the *down-stream section.* The late Professor Weisbach, of the mining-school of Freiberg, Saxony, was one of the most noted experimenters in this respect, and will be frequently quoted.

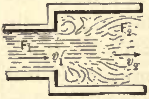

FIG. 576.

523. Loss of Head Due to Sudden (i.e., Square-edged) Enlargement. Borda's Formula.—Fig. 576. An eddy is formed in the angle with consequent loss of energy. Since each particle of water of weight $= G_1$, arriving with the velocity v_1 in the small pipe, may be considered to have an *impact* against the base of the infinitely great and more slowly moving column of water in the large pipe, and, after the impact, moves on with the same velocity, v_2, as that column, just as occurs in *inelastic direct central impact* (§ 60), we may find the energy lost by this particle on account of the impact by eq. (1) of § 138, in which, putting $M_1 = G_1 \div g$, and $M_2 = G_2 \div g =$ mass of infinitely great body of water in the large pipe, so that $M_2 = \infty$, we have

$$\text{Energy lost by particle} = G_1 \frac{(v_1 - v_2)^2}{2g}, \quad . \quad . \quad (1)$$

and the corresponding

$$\textit{Loss of head} = \frac{(v_1 - v_2)^2}{2g},$$

which, since $F_1v_1 = F_2v_2$, may be written

$$\textit{Loss of head in sudden enlargement} = \left[\frac{F_2}{F_1} - 1\right]^2 \frac{v_2^2}{2g}. \quad . \quad (2)$$

That is, the coefficient ζ for a sudden enlargement is

$$\zeta = \left(\frac{F_2}{F_1} - 1\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

F_1 and F_2 are the respective sectional areas of the pipes. Eq. (2) is *Borda's Formula.*

NOTE.—Practically, the flow cannot always be maintained with full sections. In any case, if we *assume* the pipes to be running full (once started so), and on that assumption compute the internal pressure at F_1, and find it to be zero or negative, the assumption is incorrect. That is, unless there is some pressure at F_1 the water will not swell out laterally to fill the large pipe.

EXAMPLE.—Fig. 577. In the short tube AB containing a sudden enlargement, we have given $F_2 = F_m = 6$ sq. inches, $F_1 = 4$ sq. inches, and $h = 9$ feet. Required the velocity of the jet at m (in the air, so that $p_m \div \gamma = b = 34$ ft.), if the only loss of head considered is that due to the sudden enlargement (skin-friction neglected, as the tube is short; the reservoir entrance has *rounded corners*). Applying Bernoulli's Theorem to m as down-stream section, and n in reservoir surface as up-stream position (datum level at m), we have

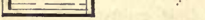

FIG. 577.

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h - \zeta\frac{v_2^2}{2g}. \quad . \quad . \quad . \quad (4)$$

But, here, $v_2 = v_m$;

$$\therefore (1 + \zeta)\frac{v_m^2}{2g} = h. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From eq. (3) we have

$$\zeta = (\tfrac{6}{4} - 1)^2 = 0.25,$$

and finally (ft., lb., sec.)

$$v_m = \sqrt{\frac{1}{1.25}}\sqrt{2 \times 32.2 \times 9} = 0.895\sqrt{2 \times 32.2 \times 9}$$

$$= 21.55 \text{ ft. per sec.}$$

(The factor 0.895 might be called a *coefficient of velocity* for this case.) Hence the volume of flow per second is

$$Q = F_m v_m = \tfrac{6}{144} \times 21.55 = 0.898 \text{ cub. ft. per sec.}$$

We have so far assumed that the water fills both parts of the tube, i.e., that the pressure p_1, at F_1, is greater than zero (see foregoing note). To verify this assumption, we compute p_1 by applying Bernoulli's Theorem to the centre of F_1 as down-stream position and datum plane, and n as up-stream position, with no loss of head between, and obtain

$$\frac{v_1^2}{2g} + \frac{p_1}{\gamma} + 0 = 0 + b + h - 0. \quad . \quad . \quad . \quad . \qquad (6)$$

But since $F_1 v_1 = F_2 v_2$, we have

$$v_1^2 = (\tfrac{6}{4})^2 v_2^2 = (\tfrac{6}{4})^2 v_m^2,$$

and hence the pressure-head at F_1 (substituting from equations above) is

$$\frac{p_1}{\gamma} = b + h - (\tfrac{6}{4})^2 \cdot \frac{h}{1 + \zeta} = 34 + 9 - \frac{9}{4} \cdot \frac{9}{1 + .25} = 27 \text{ feet},$$

and $\therefore p_1 = \frac{27}{34}$ of $14.7 = 11.6$ lbs. per sq. inch, which is greater than zero; hence efflux with the tube full in both parts can be maintained under 9 ft. head.

If, with F_1 and F_2 as before (and $\therefore \zeta$), we put $p_1 = 0$, and solve for h, we obtain $h = 42.5$ ft. as the maximum head under which efflux with the large portion full can be secured.

524. Short Pipe, Square-edged Internally.—This case, already

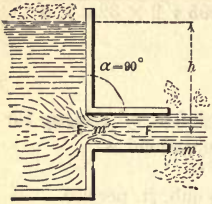

FIG. 578.

treated in §§ 507 and 515 (see Fig. 578; a repetition of 560), presents a loss of head due to the sudden enlargement from the contracted section at m' (whose sectional area may be put $= CF$, C being an unknown coefficient, or ratio, of contraction) to the full section F of the pipe. From § 515 we know that the loss of head due to the short pipe is $h_r = \zeta_E \frac{v_m^2}{2g}$ (for $\alpha = 90°$), in which $\zeta_E = 0.505$; while from Borda's Formula, § 523, we have also $\zeta_E = \left[\frac{F}{CF} - 1\right]^2$. Equating these, we find the coefficient of internal contraction at m' to be

$$C = \frac{1}{1 + \sqrt{\zeta_E}} = \frac{1}{1 + \sqrt{.505}} = 0.584,$$

or about 0.60 (compare with $C = .64$ for thin-plate contraction, § 495). It is probably somewhat larger than this (.584), since a small part of the loss of head, h_r, is due to friction at the corners and against the sides of the pipe.

By a method similar to that pursued in the example of § 523, we may show that unless h is less than 40 feet, about, the tube cannot be kept full, the discharge being as in Fig. 551. If the efflux takes place into a "partial vacuum," this limiting value of h is still smaller. Weisbach's experiments confirm these statements (but those in the C. U. Hyd. Lab. seem to indicate that the limiting value for h in the first case is about 50 ft.).

525. Diaphragm in a Cylindrical Pipe.—Fig. 579. The diaphragm being of "thin plate," let the circular opening in it (concentric with the pipe) have an area $= F$, while the sectional area of pipe $= F_2$. Beyond F, the stream contracts to a section of area $= CF = F_1$, *in enlarging*

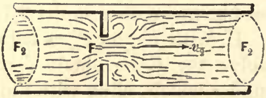

FIG. 579.

from which to fill the section F_2, of pipe, a loss of head occurs which by Borda's Formula, § 523, is

$$h_r = \zeta \frac{v_2^2}{2g} = \left(\frac{F_2}{F_1} - 1\right)^2 \frac{v_2^2}{2g},$$

where v_2 is the velocity in the pipe (*supposed full*). Of course F_1 (or CF) depends on F; but since experiments are necessary in any event, it is just as well to give the values of ζ itself, as determined by Weisbach's experiments, viz.:

For $\frac{F}{F_2} = .10$	.20	.30	.40	.50	.60	.70	.80	.90	1.00
$\zeta = 226.$	48.	17.5	7.8	3.7	1.8	.8	.3	.06	0.00

By internal lateral filling, Fig. 580, the change of section may be made gradual and eddying prevented; and then but little loss of head (and therefore little loss of energy) occurs, besides the slight amount due to skin-friction along this small surface. On p. 467 of the article *Hydromechanics* in the *Encyclopœdia Britannica* may be found an account of experiments by Mr. Froude, illustrating this fact.

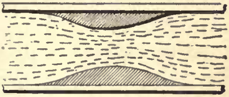

FIG. 580.

526. "The Venturi Water-meter."—The invention bearing this name was made by Mr. Clemens Herschel (see *Trans. Am. Soc. Civ. Engineers*, for November 1887), and may be described as a portion of pipe in which a gradual narrowing of section is immediately succeeded by a more gradual enlargement, as in Fig. 580; but the dimensions are more extreme. During the flow the piezometer-heights are observed at the three positions r, n, and m (see below), and the rate of discharge may be computed as follows: Referring to Fig. 580, let us denote by r the (up-stream) position where the narrowing of the pipe begins, and by m that where the enlargement ends, while n refers to the narrowest section. $F_m = F_r$.

Applying Bernoulli's Theorem to sections r and n, assuming no loss of head between, we have, as the principle of the apparatus,

$$\frac{p_r}{\gamma}+\frac{v_r^2}{2g}=\frac{p_n}{\gamma}+\frac{v_n^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence, since $F_r v_r = F_n v_n$,

$$v_n = \sqrt{\frac{1}{1-\left(\frac{F_n}{F_r}\right)^2}} \cdot \sqrt{2g\left(\frac{p_r}{\gamma}-\frac{p_n}{\gamma}\right)} = \phi\sqrt{2g\left(\frac{p_r}{\gamma}-\frac{p_n}{\gamma}\right)}, \quad (2)$$

in which ϕ represents the first radical factor. ϕ should differ but little from unity with $\frac{F_n}{F_r}$ small (and such was found to be the case by experiment). Its theoretical value is constant and greater than unity. In the actual use of the instrument the $\frac{p_r}{\gamma}$ and $\frac{p_n}{\gamma}$ are inferred from the observed piezometer-heights y_r and y_n (since $\frac{p_r}{\gamma} = y_r + b$, and $\frac{p_n}{\gamma} = y_n + b$, b being $= 34$ ft.), and then the quantity flowing per time-unit computed, from $Q = F_n v_n$, v_n having been obtained from eq. (2). This process gives a value of Q about four per cent in excess of the truth, according to the second set of experiments mentioned below, if $v_n = 35$ ft. per sec.; but only one per cent excess with $v_n = 5$ or 6 ft. per sec.

Experiments were made by Mr. Herschel on two meters of this kind, in each of which F_n was only one ninth of F_r, a ratio so extreme that the loss of head due to passage through the instrument is considerable. E.g., with the smaller apparatus, in which the diameter at n was 4 in., the loss of head between r and m was 10 or 11 ft., when the velocity through n was 50 ft. per sec., those at other velocities being roughly proportional to the square of the velocity. In the larger instrument d_n was 3 ft., and the loss of head between r and m was much more nearly proportional to the square of the velocity than in the smaller. (E.g., with $v_n = 34.56$ ft. per sec. the loss of head was 2.07 ft., while with $v_n = 16.96$ ft. per sec. it was 0.49 ft.) The angle of divergence was much smaller in these meters than that in Fig. 580.

527. Sudden Diminution of Cross-section, Square Edges.—Fig. 581. Here, again, the resistance is due to the sudden enlargement from the contracted section to the full section F_2 of the small pipe, so that in the loss of head, by Borda's formula,

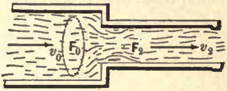

FIG. 581.

$$h_r = \zeta \frac{v_2^2}{2g} = \left[\frac{F_2}{F_1} - 1\right]^2 \frac{v_2^2}{2g}, \quad . \quad . \quad . \quad . \quad (1)$$

the coefficient

$$\zeta = \left(\frac{F_2}{F_1} - 1\right)^2 = \left(\frac{F_2}{CF_2} - 1\right)^2 = \left(\frac{1}{C} - 1\right)^2 . \quad . \quad . \quad (2)$$

depends on the coefficient of contraction C; but this latter is influenced by the ratio of F_2 to F_0, the sectional area of the larger pipe, C being about .60 when F_0 is very large (i.e., when the small pipe issues directly from a large reservoir so that $F_2 : F_0$ practically $= 0$). For other values of this ratio Weisbach gives the following table for C, from his own experiments:

For $F_2 : F_0 =$ .10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
$C =$ .624	.632	.643	.659	.681	.712	.755	.813	.892	1.00

C being found, we compute ζ from eq. (2) for use in eq. (1).

528. Elbows.—The internal disturbance caused by an elbow, Fig. 582 (pipe full, both sides of elbow), occasions a loss of head

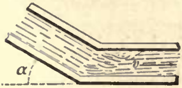

FIG. 582.

$$h_r = \zeta \frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad (1)$$

in which, according to Weisbach's experiments with tubes 3 centims., i.e. 1.2 in., in diameter, we may put

For $\alpha =$	20°	40°	60°	80°	90°	100°	110°	120°	130°	140°
$\zeta =$	.046	.139	.364	.740	.984	1.26	1.556	1.86	2.16	2.43

computed from the empirical formula;

$$\zeta = .9457 \sin^2 \tfrac{1}{2}\alpha + 2.047 \sin^4 \tfrac{1}{2}\alpha ;$$

v is the velocity in pipe; α as in figure. For larger pipes ζ would probably be somewhat smaller; and *vice versâ.*

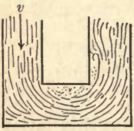

FIG. 583.

If the elbow is immediately succeeded by another in the same plane and turning the same way, Fig. 583, the loss of head is not materially increased, since the eddying takes place chiefly in the further branch of the second elbow; but if it turns in the reverse direction, Fig. 584, but still in the same plane, the total loss of head is double that of one elbow; while if the plane of the second is ┐ to that of the first, the total loss of head is $1\frac{1}{2}$ times that of one alone. (Weisbach.)

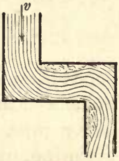

FIG. 584.

529. Bends in Pipes of Circular Section.—Fig. 585. Weisbach bases the following empirical formula for ζ, the coefficient of resistance of a quadrant bend in a pipe of circular section, on his own experiments and some of Dubuat's, viz.:

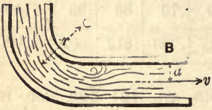

FIG. 585.

$$\zeta = 0.131 + 1.847 \left(\frac{a}{r}\right)^{\frac{7}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for use in

$$h_r \doteq \zeta \frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $a =$ radius of pipe, $r =$ radius of bend (to centre of pipe), and $v =$ velocity in pipe; $h_r =$ loss of head due to bend.

It is understood that the portion BC of the pipe is kept full by the flow; which, however, may not be practicable unless BC is more than three or four times as long as wide, and is full at the outset. A semicircular bend occasions about the same loss of head as a quadrant bend, but two quadrants forming a reverse curve in the same plane, Fig. 586, occasion a double loss. By enlarging the pipe at the bend, or providing internal thin partitions parallel to the sides, the loss of head may be considerably diminished. Weisbach gives the following table computed from eq. (1), but does not state the absolute size of the pipes.

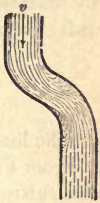

FIG. 586.

For $\frac{a}{r} = .10$	.20	.30	.40	.50	.60	.70	.80	.90	1.0
$\zeta = .131$	.138	.158	.206	.294	.440	.661	.977	1.40	1.98

Accounts of many of Weisbach's hydraulic experiments are contained in the *Civilingénieur*, vols. IX, X, and XI.

529a. Common Pipe-elbows.—Prof. L. F. Bellinger of Norwich University, Vermont, conducted a set of experiments in 1887, when a student at Cornell, on the loss of head occasioned by a common elbow (for wrought-iron pipe), whose longitudinal section is shown in Fig. 586a. The elbow served to connect at right angles two wrought-iron pipes having an internal diameter of 0.482 in.

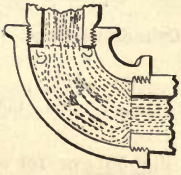

FIG. 586a.

The internal diameter of the short bend or elbow was $\frac{5}{8}$ in., and the radius of its curved circular axis (a quadrant) was $\frac{3}{4}$ in. Its internal surface was that of an ordinary rough casting.

A straight pipe of the same character and size and 14 feet long was first used, and the loss of head due to skin-friction (the only loss of head in that case) carefully determined for a range of velocities from 2 to 20 ft. per sec.

Two lengths of similar pipe were then joined by the elbow mentioned, forming a total length of 14 feet, and the total loss of head again determined through the same range of velocities. By subtraction, the loss of head due to the elbow was then easily found for each velocity, and assuming the form

$$h = \zeta \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the loss of head, ζ was computed in each case.

From Fig. 586a it is seen that the stream meets with a sudden enlargement and a sudden diminution, of section, as well as with the short bend; so that the disturbance is of a rather complex nature.

The principal results of Prof. Bellinger's experiments, after the adjustment of the observed quantities by "least squares," were found capable of being represented fairly well by the formula

$$\zeta = 0.621 + [2^n - 1] \times 0.0376, \quad . \quad . \quad . \quad (2)$$

where n = [veloc. in pipe in ft. per sec.] ÷ 5. The following table was computed from eq. (2) (where v is in ft. per second):

$v =$	2	4	6	8	10	12	14	16	18	20
$\zeta =$	.633	.649	.670	.697	.734	.782	.845	.929	1.039	1.185

530. Valve-gates and Throttle-valves in Cylindrical Pipes.— Adopting, as usual, the form

$$h_r = \zeta \frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the loss of head due to a *valve-gate*, Fig. 587, or for a

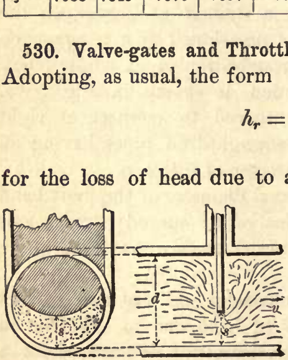

FIG. 587.

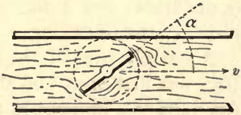

FIG. 588.

throttle-valve, Fig. 588, each in a definite position, Weisbach's experiments furnish us with a range of values of ζ in the case of these obstacles in a cylindrical pipe 1.6 inches in diameter, as follows (for meaning of s, d, and α, see figures. v is the velocity in the full section of pipe, running *full* on *both* sides.)

Valve-gate.		Throttle-valve.	
$\frac{s}{d}$	ζ	α	ζ
1.0	.00	5°	.24
$\frac{7}{8}$	.07	10°	.52
$\frac{6}{8}$	.26	15°	.90
$\frac{5}{8}$	.81	20°	1.54
$\frac{4}{8}$	2.06	25°	2.51
$\frac{3}{8}$	5.52	30°	3.91
$\frac{2}{8}$	17.00	35°	6.22
$\frac{1}{8}$	97.8	40°	10.8
		45°	18.7
		50°	32.6
		55°	58.8
		60°	118.0
		65°	256.0
		70°	751.

531. Examples involving Divers Losses of Head.—We here suppose, as before, that the pipes are full during the flow. Practically, provision must be made for the escape of the air which collects at the high points. If this air is at a tension greater than one atmosphere, automatic air-valves will serve to provide for its escape; if less than one atmosphere, an air-pump can be used, as in the case of a siphon used at the Kansas City Water Works. (See p. 346 of the *Engineering News* for November 1887.)

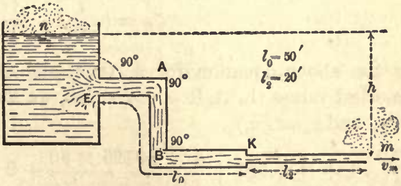

FIG. 589.

EXAMPLE 1.—Fig. 589. What head, $= h$, will be required to deliver $\frac{1}{2}$ U. S. gallon (i.e. 231 cubic inches) per second through the continuous line of pipe in the figure, containing two sizes of cylindrical pipe ($d_0 = 3$ in., and $d_2 = 1$ in.), and two 90° elbows in the larger. The flow is into the air at m, the jet being 1 in. in diameter, like the pipe. At E, $\alpha = 90°$, and the corners are not rounded; at K, also, corners not rounded. Use the ft.-lb.-sec. system of units in which $g = 32.2$.

Since $Q = \frac{1}{2}$ gal. $= \frac{1}{2}\cdot\frac{231}{1728} = .0668$ cub. ft. per sec., and therefore the velocity of the jet

$$v_m = v_2 = Q \div \tfrac{1}{4}\pi(\tfrac{1}{12})^2 = 12.25 \text{ ft. per sec.};$$

hence the velocity in the large pipe is to be $v_0 = (\frac{1}{3})^2 v_2 = 1.36$ ft. per sec. From Bernoulli's Theorem, we have, with m as datum plane,

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h - \zeta_E\frac{v_0^2}{2g} - 4f_0\frac{l_0}{d_0}\cdot\frac{v_0^2}{2g}$$
$$- 2\,\zeta_{el.}\cdot\frac{v_0^2}{2g} - \zeta_K\frac{v_2^2}{2g} - 4f_2\frac{l_2}{d_2}\cdot\frac{v_2^2}{2g},$$

involving six separate losses of head, for each of which there is no difficulty in finding the proper ζ or f, since the velocities and dimensions are all known, by consulting preceding paragraphs. (Clean iron pipe.)

From § 515, table, for $\alpha = 90°$ we have . . . $\zeta_E = 0.505$
" § 517, for $d_0 = 3$ in., and $v_0 = 1.36$ ft. per sec., $f_0 = .00725$
" " " $d_2 = 1$ in., and $v_2 = 12.25$ " " $f_2 = .00613$
" § 528 (elbows), for $\alpha = 90°$ . . . . $\zeta_{el.} = 0.984$
" § 527, for sudden diminution at K we have
[since $F_2 \div F_0 = 1^2 \div 3^2 = 0.111$, $\therefore$ $C = 0.625$]

$$\zeta_K = \left(\frac{1}{.625} - 1\right)^2 = 0.360.$$

Solving the above equation for h, then, and substituting above numerical values (in ft.-lb.-sec.-system), we have (noting that $v_m = v_2$, and $v_0 = \frac{1}{9}v_2$)

$$h = \frac{(12.25)^2}{64.4}\left[1 + (\tfrac{1}{9})^2\left(.505 + \frac{4 \times .00725 \times 50}{\frac{1}{4}} + 2 \times .984\right)\right.$$
$$\left. + .360 + \frac{4 \times .00613 \times 20}{\frac{1}{12}}\right];$$

i.e.,

$$h = \frac{(12.25)^2}{64.4}\left[1 + (.00623 + .07160 + .0243) + (.360 + 5.8848\right];$$

$$\therefore\ h = 2.323 \times 7.3469 = 17.09 \text{ ft.—}\mathit{Ans.}$$

It is here noticeable how small are the losses of head in the large pipe, the principal reason of this being that the *velocity* in it is so small ($v_0 =$ only 1.36 ft. per sec.), and that in general losses of head depend on the *square* of the velocity (nearly).

In other words, the large pipe approximates to being a reservoir in itself.

With no resistances a head $h = v_m^2 \div 2g = 2.32$ ft. would be sufficient.

Example 2.—Fig. 590. With the valve-gate V half raised (i.e., $s = \frac{1}{2}d$ in Fig. 587), required the volume delivered per second through the *clean* pipe here shown. The jet issues from a short straight pipe, or nozzle (of diameter $d_2 = 1\frac{1}{2}$ in.) inserted in the end of the larger pipe, with the inner corners *not rounded*. Dimensions as in figure. Radius of each bend $= r = 2$ in. The velocity v_m of the jet in the air = velocity v_2 in the small pipe; hence the loss of head at K is

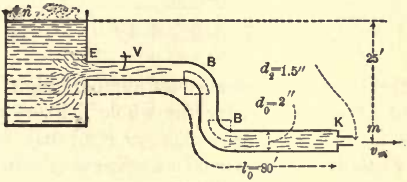

Fig. 590.

$$\zeta_K \frac{v_2^2}{2g}, \quad = \zeta_K \frac{v_m^2}{2g}.$$

Now v_m is unknown, as yet; but v_0, the velocity in the large pipe, is $= v_m \left[\frac{(\frac{3}{2})^2}{2^2}\right]$; i.e., $v_0 = \frac{9}{16}v_m$. From Bernoulli's Theorem (m as datum level) we obtain, after transposition,

$$h = \frac{v_m^2}{2g} + \zeta_E \frac{v_0^2}{2g} + \zeta_V \frac{v_0^2}{2g} + 2\zeta_B \frac{v_0^2}{2g} + 4f_0 \frac{l_0}{d_0} \frac{v_0^2}{2g} + \zeta_K \frac{v_m^2}{2g}. \quad (1)$$

Of the coefficients concerned, f_0 alone depends on the unknown velocity v_0. For the present [first approximation],

put . . . . $f_0 = .006$
From § 515, with $\alpha = 90°$, ... $\zeta_E = .505$
From § 517, valve-gate with $s = \frac{4}{8}d$, ... $\zeta_V = 2.06$
From § 529, with $a : r = 0.5$, ... $\zeta_B = 0.294$
While at K, from § 527, having

$$(F_2 : F_0) = (\tfrac{3}{2})^2 : 2^2 = \tfrac{9}{16} = 0.562;$$

we find from table, ... $C = 0.700$

and $\therefore\ \zeta_K = \left(\frac{1}{.700} - 1\right)^2 = (0.428)^2$. . . . i.e., $\zeta_K = 0.183$

Substituting in eq. (1) above, with $v_0^2 = (\frac{9}{16})^2 v_m^2$, we have

$$v_m = \sqrt{\frac{1}{1 + \frac{81}{256}\left[\zeta_E + \zeta_V + 2\zeta_B + 4f_0 \frac{l_0}{d_0}\right] + \zeta_K}}\ \sqrt{2gh}, \quad (2)$$

in which the first radical, an abstract number, might be called a *coefficient of velocity*, ϕ, for the whole delivery pipe; and also, since in this case $Q, = F_m v_m = F_2 v_2$, may be written $Q = \mu F_2 \sqrt{2gh}$, it may be named a *coefficient of efflux*, μ.

Hence

$$v_m = \sqrt{\frac{1}{1 + \frac{81}{256}\left[.505 + 2.06 + 2 \times .294 + \frac{4 \times .006 \times 80}{\frac{1}{6}}\right] + .183}}\ \sqrt{2 \times 32.2 \times 25};$$

$$\therefore\ v_m = 0.421\sqrt{2gh} = 0.421\sqrt{2 \times 32.2 \times 25} = 16.89 \text{ ft. per sec.}$$

(The .421 might be called a coefficient of velocity.) The volume delivered per second is

$$Q = \tfrac{1}{4}\pi d_2^2\, v_m = \tfrac{1}{4}\pi(\tfrac{3}{24})^2\, 16.89 = .207 \text{ cub. ft. per sec.}$$

(As the section of the jet $F_m = F_2$, that of the short pipe or nozzle, we might also say that $.421 = \mu =$ *coefficient of efflux*, for we may write $Q = \mu F_2 \sqrt{2gh}$, whence $\mu = .421$.)

532. Siphons.—In Fig. 532, § 490, the portion HN_2O is above the level, BC, of the surface of the water in the head reservoir BL, and yet under proper conditions a steady flow can be maintained with all parts of the pipe full of water, including HN_2O. If the atmosphere exerted no pressure, this would be impossible; but its average tension of 14.7 lbs. per sq. inch is equivalent to an additional depth of nearly 34 feet of water placed upon BC. With no flow, or a very small velocity, the pipe may be kept full if N_2 is not more than 33 or 34 feet above BC; but the greater v_2, the velocity of flow at N_2, and the greater and more numerous the losses of head between L and N_2, the less must be the height of N_2 above BC for a steady flow.

The analytical criterion as to whether a flow can be maintained or not, supposing the pipe completely filled at the outset, is that the internal pressure must be > 0 at all parts of the pipe. If on the supposition of a flow through a pipe of given design the pressure p is found < 0, i.e. negative, at any point [N_2 being the important section for test] the supposition is inadmissible, and the design must be altered.

For example, Fig. 532, suppose LN_2N_4 to be a long pipe of uniform section (diameter $= d$, and length $= l$), and that under the assumption of filled sections we have computed v_4, the velocity of the jet at N_4; i.e.,

$$v_4 = \sqrt{\frac{1}{1 + \zeta_L + 4f\frac{l}{d}}}\sqrt{2gh}. \quad . \quad . \quad . \quad (1)$$

To test the supposition, apply Bernoulli's Theorem to the surface BC and the point N_2 where the pressure is p_2, velocity $v_2 (= v_4$, since we have supposed a uniform section for whole pipe), and height above $BC = h_2$. Also, let length of pipe $LN_1HN_2 = l_2$. Whence we have

$$\frac{v_2^2}{2g} + \frac{p_2}{\gamma} + h_2 = 0 + \frac{p_a}{\gamma}(= b) + 0 - \zeta_L\frac{v_2^2}{2g} - 4f\frac{l_2}{d}\frac{v_2^2}{2g}. \quad . \quad (2)$$

[BC being datum plane.]

Solving for $\frac{p_2}{\gamma}$, we have

$$\frac{p_2}{\gamma} = 34 \text{ feet} - \left[h_2 + \frac{v_2^2}{2g}\left(1 + \zeta_L + 4f\frac{l_2}{d}\right)\right]. \quad . \quad . \qquad (3)$$

We note, then, that for p_2 to be > 0,

$$h_2 \textit{ must be} < \left[34 \textit{ feet} - \frac{v_2^2}{2g} - \zeta_L\frac{v_2^2}{2g} - 4f\frac{l}{d}\frac{v_2^2}{2g}\right]. \quad . \quad . \qquad (4)$$

In the practical working of a siphon it is found that atmospheric air, previously dissolved in the water, gradually collects at N_2, the highest point, during the flow and finally, if not removed, causes the latter to cease. See reference below.

One device for removing the air consists in first allowing it to collect in a chamber in communication with the pipe beneath. This communication is closed by a stop-cock after the water in it has been completely displaced by air. Another stop-cock, above, being now opened, water is poured in to replace the air, which now escapes. Then the upper stop-cock is shut and the lower one opened. The same operation is again necessary, after some hours.

On p. 346 of the *Engineering News* of November 1887 may be found an account of a siphon which has been employed since 1875 in connection with the water-works at Kansas City. It is 1350 ft. long, and transmits water from the river to the artificial "well" from which the pumping engines draw their supply. At the highest point, which is 16 ft. above low-water level of the river, is placed a "vacuum chamber" in which the air collects under a low tension corresponding to the height, and a pump is kept constantly at work to remove the air and prevent the "breaking" of the (partial) vacuum. The diameter of the pipe is 24 in., and the extremity in the "well" dips 5 ft. below the level of low water. See Trautwine's Pocket-book, for an account of Maj. Crozet's Siphon.

532a. Branching Pipes.—If the flow of water in a pipe is caused to divide and pass into two others having a common junction with the first, or *vice versâ*; or if lateral pipes lead out of a main pipe, the problem presented may be very complicated. As a comparatively simple instance, let us suppose that a pipe of diameter d and length l leads out of a reservoir, and at its extremity is joined to two others of diameters d_1 and d_2 and lengths l_1 and l_2 respectively, and that the further extremities of the latter discharge into the air without nozzles under heads h_1 and h_2 below the reservoir surface. Call these two pipes Nos. 1 and 2. That is, the system forms a Y in plan.

Assuming that all entrances and junctions are smoothly rounded, so that all loss of head is due to skin-friction, it is required to find the three velocities of flow, v, v_1, and v_2, in the respective pipes. First applying Bernoulli's Theorem to a stream-line from the reservoir surface through the main pipe to the jet at the discharging end of pipe No. 1, we have

$$\frac{v_1^2}{2g} = h_1 - 4f\frac{l}{d}\cdot\frac{v^2}{2g} - 4f\frac{l_1}{d_1}\cdot\frac{v_1^2}{2g}; \quad . \quad . \quad . \quad (1)$$

and similarly, dealing with a stream-line through the main pipe and No. 2,

$$\frac{v_2^2}{2g} = h_2 - 4f\frac{l}{d}\cdot\frac{v^2}{2g} - 4f\frac{l_2}{d_2}\cdot\frac{v_2^2}{2g}; \quad . \quad . \quad . \quad (2)$$

while the equation of continuity for this case is

$$\tfrac{1}{4}\pi d^2 v = \tfrac{1}{4}\pi d_1^2 v_1 + \tfrac{1}{4}\pi d_2^2 v_2. \quad . \quad . \quad . \quad . \quad (3)$$

From these three equations, assuming f the same in all pipes as a first approximation, we can find the three velocities (best by numerical trial, perhaps); and then the volume of discharge of the system per unit of time

$$Q = \tfrac{1}{4}\pi d^2 v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

533. Time of Emptying Vertical Prismatic Vessels (or Inclined Prisms if Bottom is Horizontal) under Variable Head.

CASE I. *Through an orifice or short pipe in the bottom and opening into the air.*—Fig. 591. As the upper free surface of area $= F'$, sinks, F' *remains constant.* Let $z =$ head of water at any stage of the emptying; it $= z_0$ at the outset, and $= 0$ when the vessel is empty. At any instant, Q, the *rate of discharge* (= volume per time-unit) depends on z and is

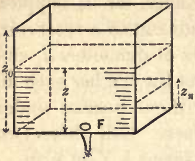

FIG. 591.

$$Q = \mu F\sqrt{2gz}, \quad . \quad . \quad . \quad (1)$$

where $\mu =$ *coefficient of efflux* $= \phi C =$ *coefficient of velocity* $\times$ *coefficient of contraction* [see § 495, eq. (3)]. We here suppose F' so large compared with F, the area of the orifice, that the free surface of the water in the vessel does not acquire any notable velocity at any stage, and that hence the rate of efflux is the same at any instant, as for a steady flow with head $= z$ and a zero velocity in the free surface. μ is considered constant. From (1) we have

$$dV = (\text{vol. discharged in time } dt) = Qdt = \mu F\sqrt{2gz}\, dt. \quad . \ (2$$

But this is also equal to the volume of the horizontal lamina, $F'dz$, through which the free surface has sunk in the same time dt.

$$\therefore\ -F'dz = \mu F\sqrt{2g}\, z^{\frac{1}{2}}dt;\ \therefore\ dt = \frac{-F'}{\mu F\sqrt{2g}}\, z^{-\frac{1}{2}}dz. \quad . \ (3)$$

We have written *minus* $F'dz$ because, dt being an increment, dz is a decrement. To reduce the depth from z_0 (at the start, time $= t =$ zero) to z_n, demands a time

$$\Big[_0^n t = -\frac{F'}{\mu F\sqrt{2g}}\int_{z_0}^{z_n} z^{-\frac{1}{2}}\, dz = \frac{2F'}{\mu F\sqrt{2g}}\,[z_0^{\frac{1}{2}} - z_n^{\frac{1}{2}}]; \ (4)$$

whence, by putting $z_n = 0$, we have the time necessary to empty the whole prism

$$\Big[_0^n t = \frac{2F'z_0^{\frac{1}{2}}}{\mu F\sqrt{2g}} = \frac{2F'z_0}{\mu F\sqrt{2gz_0}} = \frac{2 \times \text{volume of vessel}}{\text{initial rate of discharge}}; (5)$$

that is, to empty the vessel requires *double the time* of discharging the same amount of water if the vessel had been kept full (at constant head $= z_0 =$ altitude of prism).

To *fill* the same vessel through an orifice in the bottom, the flow through which is supplied from a body of water of infinite extent horizontally, as with the (single) canal lock of Fig. 592, will obviously require the same time as given in eq. (5) above, since the effective head z diminishes from z_0 to 0, according to the same law.

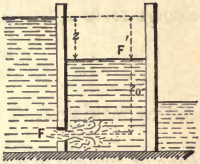

FIG. 592.

EXAMPLE.—What time will be needed to empty a parallelopipedical tank (Fig. 591) 4 ft. by 5 ft. in horizontal section and 6 ft. deep, through a stop-cock in the bottom whose coefficient of efflux when fully open is known to be $\mu = 0.640$, and whose section of discharge is a circle of diameter $= \frac{1}{2}$ in.? From given dimensions $F' = 4 \times 5 = 20$ sq. ft., while $z_0 = 6$ ft. Hence from eq. (5) (ft.-lb.-sec.)

$$\left.\begin{matrix}\textit{time of}\\ \textit{emptying}\end{matrix}\right\} = \frac{2 \times 20 \times \sqrt{6}}{0.64 \times \frac{1}{4}\pi(\frac{1}{24})^2 \sqrt{2 \times 32.2}} = \left\{\begin{matrix}13980 \text{ seconds}\\ = 3^{\text{hours}} 53^{\text{min.}} 0^{\text{sec.}}\end{matrix}\right.$$

CASE II. *Two communicating prismatic vessels. Required the time* for the water to come to a common level ON, Fig. 593, efflux taking place through a small orifice, of area $= F$, under water. At any instant the *rate of discharge* is

$$Q = \mu F \sqrt{2gz},$$

as before. $z =$ difference of level. Now if F' and F'' are the horizontal sectional areas of the two prismatic vessels (axes vertical) we have $F'x = F''y$, and hence z, which $= x + y$, $= x + (F' \div F'')x$;

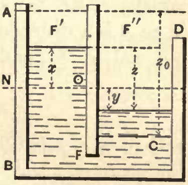

FIG. 593.

$$\therefore\ x = \frac{z}{1 + \frac{F'}{F''}}, \quad \text{and} \quad dx = \frac{dz}{1 + \frac{F'}{F''}}.$$

As before, we have

$$-F'dx = \mu F\sqrt{2g}\, z^{\frac{1}{2}}dt, \quad \text{or} \quad dt = -\frac{F'F''}{F'+F''}\frac{z^{-\frac{1}{2}}dz}{\mu F\sqrt{2g}}.$$

Hence, integrating, the time for the *difference* of level to change from z_0 to z_n

$$= \frac{2F'F''}{F'+F''}\cdot\frac{z_0^{\frac{1}{2}} - z_n^{\frac{1}{2}}}{\mu F\sqrt{2g}}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and by making $z_n = 0$ in (6), we have the

$$\textit{time of coming to a common level} = \frac{2F'F''}{F'+F''}\cdot\frac{1}{\mu F}\sqrt{\frac{z_0}{2g}}. \qquad (7)$$

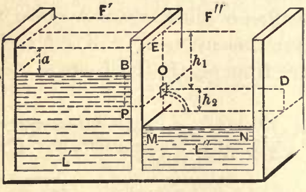

FIG. 594.

ALGEBRAIC EXAMPLE.—In the double lock in Fig. 594, let L' be full, while in L'' the water stands at a level MN the same as that of the tail-water. F' and F'' are the horizontal sectional areas of the prismatic locks. Let the orifice, O, between them, be at a depth $= h_1$ below the initial level KE of L', and a height $= h_2$ *above* that, MN, of L''. The orifice at O, area $= F$, being opened, efflux from L' begins *into the air*. and the level of L'' is gradually raised from MN to OD, while that of L' sinks from KE to AB a distance $= a$, computed from the relation vol. $F'a =$ vol. $F''h_2$, and the time occupied is [eq. (4)]

$$t_1 = \frac{2F'}{\mu F\sqrt{2g}}[\sqrt{h_1} - \sqrt{h_1 - a}]. \quad . \quad . \quad . \quad . \qquad (8)$$

As soon as O is submerged, efflux takes place under water, and we have an instance of Case II. Hence the time of reaching a common level (after submersion of O) (see eq. 7) is

$$t_2 = \frac{2F'F''}{\mu F(F'+F'')}\sqrt{\frac{h_1 - a}{2g}}, \quad . \quad . \quad . \quad . \qquad (9)$$

and the total time is $= t_1 + t_2$, with $a = F''h_2 \div F'$.

CASE III. *Emptying a vertical prismatic vessel through a rectangular "notch" in the side, or overfall.*—Fig. 595. As before, let even the initial area ($= z_0b$) of the notch be small compared with the horizontal area F' of tank. Let $z =$ depth of lower sill of notch below level of tank surface at any instant, and $b =$ width of notch. Then, at any instant (see eq. 10, § 504),

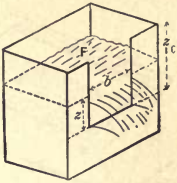

FIG. 595.

$$Rate\ of\ disch.\ (vol.) = Q = \tfrac{2}{3}\mu_0 bz\sqrt{2gz} = \tfrac{2}{3}\mu b\sqrt{2g}\,z^{\frac{3}{2}}.$$

$$\therefore \text{ vol. of disch. in } dt = \tfrac{2}{3}\mu b\sqrt{2g}\,z^{\frac{3}{2}}dt,$$

and putting this $= -F'dz =$ vol. of water lost by the tank in time dt, we have

$$dt = -\frac{3}{2}\frac{F'}{\mu b\sqrt{2g}}z^{-\frac{3}{2}}dz;$$

whence

$$\Big[_0^n t = -\frac{3}{2}\frac{F'}{\mu b\sqrt{2g}}\int_{z_0}^{z_n} z^{-\frac{3}{2}}dz = -\frac{3}{2}\frac{F'}{\mu b\sqrt{2g}}\Big[_0^n \frac{z^{-\frac{1}{2}}}{-\frac{1}{2}};$$

i.e.,

$$\Big[_0^n t = \frac{3F'}{\mu b\sqrt{2g}}\left[\frac{1}{\sqrt{z_n}} - \frac{1}{\sqrt{z_0}}\right], \quad . \quad . \quad . \quad . \quad (10)$$

as the time in which the tank surface sinks from a height z_0 above sill to a height z_n above sill. If we inquire the time t' for the water to sink to the level *of the sill of the notch* we put $z_n =$ zero, whence $t' =$ *infinity*. As explanatory of this result, note that as z diminishes not only does the velocity of flow diminish, but the available area of efflux ($= zb$) also grows less, whereas in Cases I and II the orifice of efflux remained of constant area $= F$.

Eq. (10) is applicable to the waste-weir of a large reservoir or pond.

534. Time of Emptying Vessels of Variable Horizontal Sections.—Considering regular geometrical forms first, let us take

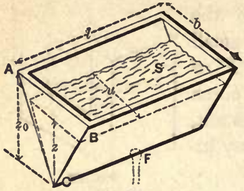

Fig. 596.

Case I. *Wedge-shaped vessel, edge horizontal and underneath, orifice F in the edge,* so that z, the variable head, is always the altitude of a triangle similar to the section ABC of the body of water when efflux begins. At any instant during the efflux the area, S, of the free surface, *variable here*, takes the place of F' in eq. (3) of § 533, whence we have,

for any case of variable free surface, $$dt = \frac{-Sz^{-\frac{1}{2}}dz}{\mu F\sqrt{2g}}. \quad . \quad (11)$$

In the present case $S = ul$, and from similar triangles

$$u : z :: b : z_0;$$

whence

$$S = blz \div z_0,$$

and

$$dt = \frac{-blz^{\frac{1}{2}}dz}{\mu F z_0 \sqrt{2g}};$$

$$\therefore \left[t\right]_0^n = \frac{bl}{\mu F z_0 \sqrt{2g}}\int_{z_n}^{z_0} z^{\frac{1}{2}}dz = \frac{\frac{2}{3}bl}{\mu F z_0\sqrt{2g}}\left[z_0^{\frac{3}{2}} - z_n^{\frac{3}{2}}\right], \quad . \quad (12)$$

and hence the time of emptying the whole *wedge*, putting $z_n = 0$, is

$$t_0 = \frac{4}{3}\cdot\frac{\frac{1}{2}blz_0}{\mu F\sqrt{2gz_0}} = \frac{4}{3}\cdot\frac{\textit{Vol. of wedge}}{\textit{initial rate of discharge}}; \quad . \quad (13)$$

i.e., $\frac{4}{3}$ as long as to discharge the same volume of water under a constant head $= z_0$. This is equally true if the ends of the wedge are *oblique*, so long as they are parallel.

Case II. *Right segment of paraboloid of revolution.*—Fig. 597. Axis vertical. Orifice at vertex. Here the variable free surface has at any instant an area, $= S, = \pi u^2$, u being the radius of the circle and variable. From a property of the parabola

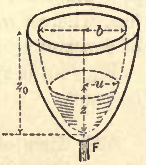

Fig. 597.

$$u^2 : b^2 :: z : z_0; \quad \therefore \quad S = \pi b^2 z \div z_0,$$

and hence, from eq. (11),

$$dt = \frac{-\pi b^2 z^{\frac{1}{2}} dz}{\mu F z_0 \sqrt{2g}};$$

$$\therefore \left[t\right]_0^n = \frac{\pi b^2}{\mu F z_0 \sqrt{2g}} \int_{z_n}^{z_0} z^{\frac{1}{2}} dz = \frac{2}{3} \frac{\pi b^2}{\mu F z_0 \sqrt{2g}} [z_0^{\frac{3}{2}} - z_n^{\frac{3}{2}}];$$

whence, putting $z_n = 0$, we have the time of emptying the whole vessel

$$t_0 = \frac{4}{3} \frac{\pi b^2 \frac{1}{2} z_0}{\mu F \sqrt{2g z_0}} = \frac{4}{3} \cdot \frac{\textit{total vol.}}{\textit{initial rate of disch.}}, \quad . \quad (14)$$

same result as for the wedge, in Case I; in fact, it applies to any vessel in which the *areas of horizontal sections vary directly with their heights above the orifice.*

CASE III. *Any pyramid or cone; vertex down; small orifice in vertex.*—Fig. 598. Let area of the base $= S_0$, at upper edge of vessel. At any stage of the flow $S =$ area of base of pyramid of water. From similar pyramids

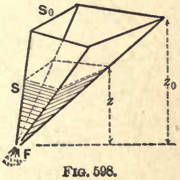

FIG. 598.

$$S_0 : S :: z_0^2 : z^2; \quad \therefore \quad S = \frac{S_0}{z_0^2} z^2,$$

and [eq. (11)]

$$dt = -\frac{S_0}{z_0^2} \frac{1}{\mu F \sqrt{2g}} z^{\frac{3}{2}} dz,$$

whence ($z_n = 0$) the time of emptying the whole vessel is

$$t_0 = \frac{S_0}{\mu F z_0^2 \sqrt{2g}} \int_0^{z_0} z^{\frac{3}{2}} dz = \frac{2}{5} \frac{S_0 z_0^{\frac{5}{2}}}{\mu F z_0^2 \sqrt{2g}};$$

or,

$$t_0 = \frac{6}{5} \cdot \frac{\textit{Total volume}}{\textit{initial rate of disch.}} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

CASE IV. *Sphere.*—Similarly, we may show that to empty a sphere, of radius $= r$, through a small orifice, of area $= F$, in lowest part, the necessary time is

$$t_0 = \frac{16}{15} \cdot \frac{\pi r^3}{\mu F \sqrt{gr}} = \frac{8}{5} \cdot \frac{Vol.}{init.\ rate\ of\ disch.}$$

535. Time of Emptying an Obelisk-shaped Vessel.—(An obelisk may be defined as a solid of six plane faces, two of which are rectangles in parallel planes and with sides respectively parallel, the others trapezoids; a frustum of a pyramid is a particular case.)

A volume of this shape is of common occurrence; see Fig. 599. Let the altitude $= h$, the two rectangular faces being horizontal, with dimensions as in figure. By drawing through F, G, and H right lines parallel to EC, to cut the upper base, we form a rectangle $KLMC$ equal to the lower base. Produce ML to P and KL to N, and join PG and NG. We have now subdivided the solid into a parallelopiped $KLMC$-$EHGF$, a pyramid $PBNL$-G, and two wedges, viz. $APLK$-HG and $LNDM$-FG, with their edges horizontal; and may obtain the time necessary to empty the whole obelisk-volume by adding the times which would be necessary to empty the individual component volumes, separately, through the same orifice or pipe in the bottom plane EG. These have been already determined in the preceding paragraphs. The dimensions of each component volume may be expressed in terms of those of the obelisk, and all have a common altitude $= h$.

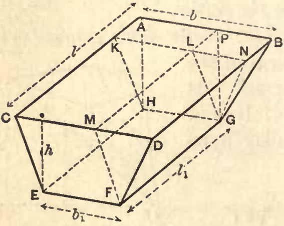

FIG. 599.

Assuming the orifice to be in the bottom, or else that the discharging end of the pipe, if such is used, is *in the plane of the bottom EG*, we have as follows, F being the area of discharge:

$$\left.\begin{array}{l}\textit{Time to empty the parallelopiped}\\ \textit{separately would be (Case I, § 533)}\end{array}\right\} \;.\;.\; t_1 = \frac{2b_1l_1}{\mu F\sqrt{2g}}\sqrt{h}. \quad (1)$$

$$\left.\begin{array}{l}\textit{Time to empty the two}\\ \quad\textit{wedges separately}\\ \quad\textit{(Case I, § 534)}\end{array}\right\} t_2 = \frac{2}{3}\cdot\frac{b_1(l-l_1)+l_1(b-b_1)}{\mu F\sqrt{2g}}\sqrt{h}\;. \quad (2)$$

$$\left.\begin{array}{l}\textit{For the pyramid}\\ \textit{(Case III, § 534)}\end{array}\right\} \;.\;.\;.\; t_3 = \frac{2}{5}\cdot\frac{(l-l_1)(b-b_1)}{\mu F\sqrt{2g}}\sqrt{h}. \;\;.\;\;.\quad (3)$$

Hence to empty the whole reservoir we have

$$t = t_1 + t_2 + t_3;$$ Be careful

i.e.,

$$t = [3bl + 8b_1l_1 + 2bl_1 + 2b_1l]\frac{2\sqrt{h}}{15\mu F\sqrt{2g}}. \;\;.\;\;.\;\;.\quad (4)$$

Example.—Let a reservoir of above form, and with $b = 50$ ft., $l = 60$ ft., $b_1 = 10$ ft., $l_1 = 20$ ft., and depth of water $h = 16$ ft., be emptied through a straight iron pipe, horizontal, and leaving the side of the reservoir *close to the bottom*, at an angle $\alpha = 36°$ with the inner plane of side. The pipe is 80 ft. long and 4 inches in internal diameter; and of clean surface. The jet issues directly from this pipe into the air, and hence $F = \frac{1}{4}\pi(\frac{1}{3})^2$ sq. feet. To find μ, the "coefficient of efflux" $(= \phi$, the coefficient of velocity in this case, since there is no contraction at discharge orifice), we use eq. (4) (the first radical) of § 518, with f approx. $= .006$, and obtain

$$\mu = \phi = \sqrt{\frac{1}{1+\zeta_E+4f\frac{l}{d}}} = \sqrt{\frac{1}{1+.896+\frac{4\times .006\times 80}{\frac{1}{3}}}} = 0.361.$$

(N.B. Since the velocity in the pipe diminishes from a value

$$v = .361\sqrt{2g\times 16} = 11.6 \text{ ft. per sec.}$$

at the beginning of the flow to $v =$ zero at the close, $f = .006$ is a reasonably approximate average with which to compute the average ϕ above; see § 517.

Hence from eq. (4) of this paragraph (ft.-lb.-sec. system)

$$t = \frac{[3 \times 50 \times 60 + 8 \times 10 \times 20 + 2(50 \times 20 + 10 \times 60)]2\sqrt{16}}{15 \times 0.361 \times \frac{\pi}{4}\left(\frac{1}{3}\right)^2 \sqrt{2 \times 32.2}}$$

$$= 29110 \text{ sec.} = 8 \text{ hrs. } 5 \text{ min.} \left\{ \begin{matrix}\text{Probably within 2 or 3\% of} \\ \text{the truth.}\end{matrix} \right.$$

536. Time of Emptying Reservoirs of Irregular Shape. Simpson's Rule.—From eq. (11), § 534, we have, for the time in which the free surface of water in a vessel of any shape whatever sinks through a vertical distance $= dz$,

$$dt = \frac{-Sz^{-\frac{1}{2}}dz}{\mu F\sqrt{2g}}, \text{ whence } \left[\text{time}\right]_{z=z_0}^{z=z_n} = \frac{1}{\mu F\sqrt{2g}}\int_{z_n}^{z_0} Sz^{-\frac{1}{2}}dz, \quad . \; . \quad (1)$$

where S is the variable area of the free surface at any instant, and z the head of water at the same instant, efflux proceeding through a small orifice (or extremity of pipe) of area $= F$. If S can be expressed in terms of z, we can integrate eq. (1) (i.e., provided that $Sz^{-\frac{1}{2}}$ has a known anti-derivative); but if not, the vessel or reservoir being irregular in form, as in Fig. 600 (which shows a pond whose bottom has been accurately surveyed, so that we know the value of S for any stage of the emptying), we can still get an approximate solution by using Simpson's Rule for approximate integration. Accordingly, if we inquire the time in which the surface will sink from 0 to the entrance E of the pipe in Fig. 600 (any point n; at E. or short of that), we divide the *vertical distance* from 0 to n (4 in this figure) into an even number of equal parts, and from the known form of the pond compute the area S corresponding to each point of division, calling them S_0, S_1, etc. Then the required time is approximately

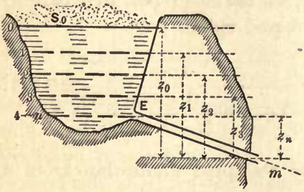

FIG. 600.

$$\Big[_0^n t = \frac{z_0 - z_n}{3\mu F\sqrt{2g}\,n}\Big[\frac{S_0}{z_0^{\frac{1}{2}}} + 4\Big(\frac{S_1}{z_1^{\frac{1}{2}}} + \frac{S_3}{z_3^{\frac{1}{2}}} + \dots \frac{S_{n-1}}{z_{n-1}^{\frac{1}{2}}}\Big)$$

$$+ 2\Big(\frac{S_2}{z_2^{\frac{1}{2}}} + \frac{S_4}{z_4^{\frac{1}{2}}} + \dots \frac{S_{n-2}}{z_{n-2}^{\frac{1}{2}}}\Big) + \frac{S_n}{z_n^{\frac{1}{2}}}\Big].$$

Example.—Fig. 600. Suppose we have a pipe *Em* of the same design as in the example of § 535, and an initial head of $z_0 = 16$ ft., so that the same value of $\mu, = .361$, may be used. Let $z_n - z_0 = 8$ feet, and divide this interval (of 8 ft.) into four equal vertical spaces of 2 ft. each. If at the respective points of division we find from a previous survey that $S_0 = 400000$ sq. ft., $S_1 = 320000$ sq. ft., $S_2 = 270000$ sq. ft., $S_3 = 210000$ sq. ft., and $S_4 = 180000$ sq. ft.; while $n = 4$, $\mu = .361$, and the area $F = \frac{1}{4}\pi(\frac{1}{3})^2 = .0873$ sq. ft., we obtain (ft., lb., sec.)

$$\Big[_0^4 t = \frac{16 - 8}{0.361 \times .0873\,\sqrt{2 \times 32.2} \times 3 \times 4}\Big[\frac{400000}{\sqrt{16}} + \frac{4 \times 320000}{\sqrt{14}}$$

$$+ \frac{2 \times 270000}{\sqrt{12}} + \frac{4 \times 210000}{\sqrt{10}} + \frac{180000}{\sqrt{8}}\Big] \begin{array}{l} = 2444000 \text{ sec.} \\ = 28^{\text{d.}}\ 6^{\text{h.}}\ 53^{\text{m.}}\ 20^{\text{s.}} \end{array}$$

The volume discharged, *V*, may also be found by Simpson's Rule, thus: Since each infinitely small horizontal lamina has a volume

$$dV = -Sdz, \quad \therefore \quad \Big[_0^n V = \int_n^0 Sdz,$$

or, approximately,

$$\Big[_0^n V = \frac{z_0 - z_n}{3n}\Big[S_0 + 4S_1 + 2S_2 + 4S_3 + \dots + S_n\Big].$$

Hence with $n = 4$ we have (ft., lb., sec.)

$$\Big[_0^n V = \frac{16 - 8}{3 \times 4}\Big[400000 + 4\left\{\begin{array}{l} 320000 \\ + 210000 \end{array}\right\} + 2 \times 270000$$

$$+ 180000\Big] = 2{,}160{,}000 \text{ cub. ft.}$$

537. Volume of Irregular Reservoir Determined by Observing Progress of Emptying.—Transforming eq. (11), § 534, we have

$$Sdz = -\mu F\sqrt{2g}\,z^{\frac{1}{2}}dt.$$

But Sdz is the infinitely small volume dV of water lost by the reservoir in the time dt, so that the volume of the reservoir between the initial and final (0 and n) positions of the horizontal free surface (at beginning and end of the time t_n) may be written

$$\Big[_0^n V = \mu F\sqrt{2g}\int_0^{t_n} z^{\frac{1}{2}}dt. \quad . \quad . \quad . \quad . \quad (1)$$

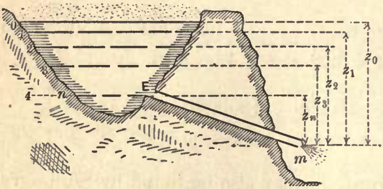

FIG. 601.

This can be integrated approximately by Simpson's Rule, if the whole *time of emptying*, $= t_n$, be divided into an even number of equal parts, and the values z_0, z_1, z_2, etc., of the head of water noted at these *equal intervals of time* (not of vertical height). The corresponding surface planes will not be equidistant, in general. Whence for the particular case when $n = 4$ (see Fig. 601)

$$\Big[_0^4 V = \frac{\mu F\sqrt{2g}\,(t_n - 0)}{3 \times 4}\,[z_0^{\frac{1}{2}} + 4\,(z_1^{\frac{1}{2}} + z_3^{\frac{1}{2}}) + 2z_2^{\frac{1}{2}} + z_4^{\frac{1}{2}}]. \;.\;.\; (2)$$

CHAPTER VII.

HYDRODYNAMICS (*Continued*)—STEADY FLOW OF WATER IN OPEN CHANNELS.

538. Nomenclature.—Fig. 602. When water flows in an open channel, as in rivers, canals, mill-races, water-courses, ditches, etc., the bed and banks being rigid, the upper surface is free to conform in shape to the dynamic conditions of each case, which therefore regulate to that extent the shape of the cross-section.

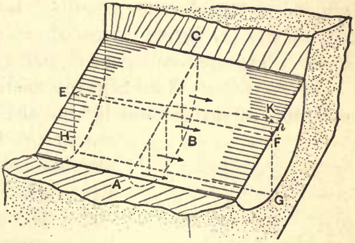

FIG. 602.

In the vertical transverse section AC in figure, the line AC is called the *air-profile* (usually to be considered horizontal and straight), while the line ABC, or profile of the bed and banks, is called the *wetted perimeter*. It is evident that the ratio of the wetted perimeter to the whole perimeter, though never $< \frac{1}{2}$, varies with the form of the transverse section.

In a longitudinal section of the stream, $EFGH$, the angle made by a surface filament EF with the horizontal is called the *slope*, and is measured by the ratio $s = h : l$, where l is the length of a portion of the filament and $h =$ the *fall*, or *vertical* distance between the two ends of that length. The angle between the horizontal and the line HG along the bottom is not necessarily equal to that of the surface, unless the portion of the stream forms a prism; i.e., the slope of the bed does not necessarily $= s =$ that of surface.

EXAMPLES.—The old Croton Aqueduct has a slope of 1.10 ft. per mile; i.e., $s = .000208$. The new aqueduct (for New

York) has a slope $s = .000132$, with a larger transverse section. For large sluggish rivers s is much smaller.

539. Velocity Measurements. — Various instruments and methods may be employed for this object, some of which are the following:

Surface-floats are small balls, or pieces of wood, etc., so colored and weighted as to be readily seen, and still but little affected by the wind. These are allowed to float with the current in different parts of the width of the stream, and the surface velocity c in each experiment computed from $c = l \div t$, where l is the distance described between parallel transverse alignments (or actual ropes where possible), whose distance apart is measured on the bank, and $t =$ the time occupied.

Double-floats. Two balls (or small kegs) of same bulk and condition of surface, one lighter, the other heavier than water, are united by a slender chain, their weights being so adjusted that the light ball, without projecting notably above the surface, buoys the other ball at any assigned depth. Fig. 603. It is assumed that the combination moves with a velocity c', equal to the arithmetic mean of the surface velocity c_0 of the stream and that, c, of the water filaments at the depth of the lower ball, which latter, c, is generally less than c_0. That is, we have

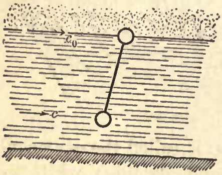

FIG. 603.

$$c' = \tfrac{1}{2}(c_0 + c) \quad \text{and} \quad \therefore \; c = 2c' - c_0 . \quad . \quad . \quad . \quad (1)$$

Hence, c_0 having been previously obtained, eq. (1) gives the velocity c at any depth of the lower ball, c' being observed.

The *floating staff* is a hollow cylindrical rod, of adjustable length, weighted to float upright with the top just visible. Its observed velocity is assumed to be an average of the velocities of all the filaments lying between the ends of the rod.

Woltmann's Mill; or Tachometer; or Current-meter, Fig. 604, consists of a small wheel with inclined floats (or of a small "screw-propeller" wheel S) held with its plane ⊥ to the current, which causes it to revolve at a speed nearly proportional to the velocity, c, of the water passing it. By a screw-gearing W on the shaft, connection is made with a counting apparatus to record the number of revolutions. Sometimes a vane B is attached, to compel the wheel to face the current. It is either held at the extremity of a pole or, by being adjustable along a vertical staff fixed in the bed, may be set at any desired depth below the surface. It is usually so designed as to be thrown in and out of gear by a cord and spring, that the time of making the indicated number of revolutions may be exactly noted.

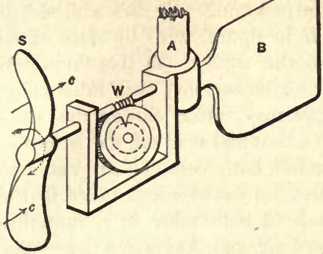

Fig. 604.

By experiments in currents of known velocities a table or formula can be constructed by which to interpret the indications of any one instrument; i.e., to find the velocity c of the current corresponding to an observed number of revolutions per minute.

A peculiar form of this instrument has been recently invented, called the *Ritchie-Haskell Direction-current Meter*, for which the following is claimed: "This meter registers electrically on dials in boat from which used, the *direction* and *velocity*, simultaneously, of any current. Can be used in river, harbor, or ocean currents."

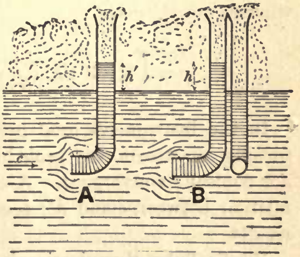

Fig. 605.

Pitot's Tube consists in principle of a vertical tube open above, while its lower end, also open, is bent horizontally upstream; see A in figure. After the oscillations have ceased, the water in the tube remains stationary with its free surface a height, h', above that of the stream, on account of the continuous impact of the current against the lower end of the column. By the addition of another vertical tube (see B in figure) with the face of its lower (open) end parallel to the current (so that the water-level in it is the same as that of the current), both tubes being provided with stop-cocks, we may, after closing the stop-cocks, lift the apparatus into a boat and read off the height h' at leisure. We may also cause both columns of water to mount, through flexible tubes, into convenient tubes in the boat by putting the upper ends of both tubes in communication with a receiver of rarefied air, and thus watch the oscillations and obtain a more accurate value of h'. [See Van Nostrand's Mag. for Mar. '78, p. 255.] Theoretically (see § 565), the thickness of the walls of the tube at the lower end being considerable, we have

$$c = \sqrt{gh'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as a relation between c, the velocity of the particles impinging on the lower end, and the static height h' (§ 565). Eq. (1) is verified fairly well by Weisbach's experiments with fine instruments, used with velocities of from 0.32 to 1.24 meters per second. Weisbach found

$$c = \left[3.54 \sqrt{h' \ (in\ meters)} \right] \text{met. per sec.},$$

whereas eq. (1) gives

$$c = \left[3.133 \sqrt{h' \ (in\ meters)} \right] \text{met. per sec.}$$

In the instruments used by Weisbach the end of the tube turning up-stream was probably straight; i.e., neither flaring nor conically convergent. A change in this respect alters the relation between c and h'; see § 565 for Pitot's and Darcy's results.

Pitot's Tube, though simple, is not so accurate as the tachometer.

The *Hydrometric Pendulum*, a rather uncertain instrument, is readily understood from Fig. 606. The side AB, of the quadrant ABC, being held vertical, the plane of the quadrant is made parallel to the current. The angle θ between the cord and the vertical depends on G, the *effective weight* (i.e., actual weight diminished by the buoyant effort) of the ball (heavier than water), and the amount of P, the impulse or horizontal pressure of the current against the latter, since the cord will take the direction of the resultant R, for equilibrium.

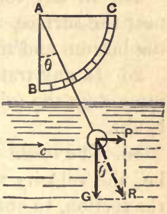

FIG. 606.

Now P (see § 572) for a ball of given size and character of surface varies (nearly) as the square of the velocity; i.e., if P' is the impulse on a given *stationary* ball, when the velocity of the current $= c'$, then for any other velocity c we have

$$P = impulse = \frac{P'}{c'^2}c^2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From this and the relation $\tan\theta = \frac{P}{G}$ we derive

$$c = \sqrt{\frac{Gc'^2}{P'}} . \sqrt{\tan\theta}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

With a given instrument and a *specified system* of units, the numerical value of the first radical may be determined as a *single quantity*, by experimenting with a known velocity and the value of θ then indicated, and may then, as a constant factor, be employed in (3) for finding the value of c for *any* observed value of θ; but the *same units* must be used as before.

540. Velocities in Different Parts of a Transverse Section.— The results of velocity-measurements made by many experimenters do not agree in supporting any very definite relation between the greatest surface velocity ($c_{0\,\max.}$) of a transverse section and the velocities at other points of the section, but establish a few general propositions:

1st. In any vertical line the velocity is a maximum quite near the surface, and diminishes from that point both toward the bottom and toward the surface.

2d. In any transverse horizontal line the velocity is a maximum near the middle of the stream, diminishing toward the banks.

3d. The *mean velocity* $= v$, of the whole transverse section, i.e., the velocity which must be multiplied by the area, F, of the section, to obtain the volume delivered per unit of time,

$$Q = Fv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is about 83 per cent of the maximum surface velocity (c_0 max.) observed when the air is still; i.e.,

$$v = 0.83 \times (c_{0\ \mathrm{max.}}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Of eight experimenters cited by Prof. Bowser, only one gives a value ($= 0.62$) differing more than .05 from .83, while others obtained the values .82, .78, .82, .80, .82, .83.

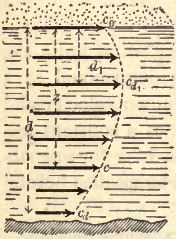

FIG. 607.

In the survey of the Mississippi River by Humphreys and Abbot, 1861, it was found that the law of variation of the velocity in any given vertical line could be fairly well represented by the ordinates of a parabola (Fig. 607) with its axis horizontal and its vertex at a distance d_1 below the surface according to the following relation, f'' being a number dependent on the force of the wind (from 0 for no wind to 10 for a hurricane):

$$d_1 = [0.317 \pm 0.06 f'']d; \quad . \quad . \quad (3)$$

where d is the total depth, and the double sign is to be taken $+$ for an up-stream, $-$ for a down-stream, wind. The following relations were also based on the results of the survey:

$$\text{(putting, for brevity, } B = 1.69 \div \sqrt{d + 1.5}\text{,)} \quad . \quad (4)$$

$$c = c_{d_1} - \sqrt{Bv}\left(\frac{z - d_1}{d}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$c_m = \tfrac{2}{3}c_{d_1} + \tfrac{1}{3}c_d + \frac{d_1}{d}\left(\tfrac{1}{3}c_0 - \tfrac{1}{3}c_d\right), \quad . \quad . \quad . \quad . \quad (6)$$

and

$$c_{\frac{1}{2}d} = c_m + \tfrac{1}{12}\sqrt{Bv}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(*These equations are not of homogeneous form*, but call for the *foot* and *second* as units.)

In (4), (5), and (6),

c = velocity at any depth z below the surface;
c_m = mean velocity in the vertical curve;
c_{d_1} = max. " " " "
$c_{\frac{1}{2}d_1}$ = " at mid-depth;
c_d = velocity at bottom;
v = mean velocity of the whole transverse section.

It was also found that the parameter of the parabola varied inversely as the square root of the mean velocity c_m of curve.

In general the bottom velocity (c_d) is somewhat more than $\frac{1}{2}$ the maximum velocity (c_{d_1}) in the same vertical. In the Mississippi the velocity at mid-depth in any vertical was found to be very nearly .96 of the surface velocity in the same vertical; this fact is important, as it simplifies the approximate gauging of a stream.

541. Gauging a Stream or River.—Where the relation (eq. (2). § 540) $v = .83\ (c_{0\,\max.})$ is not considered accurate enough for substitution in $Q = Fv$ to obtain the volume of discharge (or delivery) Q of a stream per time-unit, the transverse section may be divided into a number of subdivisions as in Fig. 608, of widths a_1, a_2, etc., and mean depths d_1, d_2, etc., and the respective mean velocities, c_1, c_2, etc., computed from measurements with current-meters; whence we may write

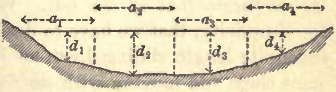

FIG. 608.

$$Q = a_1d_1c_1 + a_2d_2c_2 + a_3d_3c_3 + \text{etc.} \quad . \quad . \quad . \quad (7)$$

With a small stream or ditch, however, we may erect a vertical boarding, and allow the water to flow through a rectangular notch or overfall, Fig. 609, and after the head surface has become permanent, measure h_2 (depth of sill below the *level* surface somewhat back of boards), and b (width) and use the formulæ of § 504; see examples in that article.

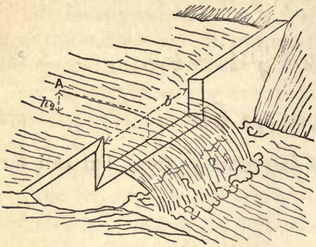

FIG. 609.

542. Uniform Motion in an Open Channel.—We shall now consider a straight stream of indefinite length in which *the flow is steady*, i.e., a *state of permanency exists*, as distinguished from a freshet or a wave. That is, the flow is steady when the water assumes fixed values of mean velocity v, and sectional area F, on passing a given point of the bed or bank; and the

Eq. of continuity . . $Q = Fv = F_0v_0 = F_1v_1 =$ constant . . (1)

holds good whether those sections are equal or not.

By *uniform motion* is meant that (the section of the bed and banks being of constant size and shape) the slope of the bed, the quantity of water (volume $= Q$) flowing per time-unit, and the extent of the wetted perimeter, are so adjusted to each other that the mean velocity of flow is the same in all transverse sections, and consequently the area and shape of the transverse section is the same at all points; and *the slope of the surface = that of the bed.* We may therefore consider, for simplicity, that we have to deal with a prism of water of indefinite length sliding down an inclined rough bed of constant slope and moving with *uniform* velocity (viz., the mean velocity v common to all the sections); that is, there is *no acceleration.* Let Fig. 610 show, free, a portion of this prism, of length $= l$, and having its bases ┐ to the bed and surface.

The hydrostatic pressures at the two ends balance each other from the identity of conditions. The only other forces having

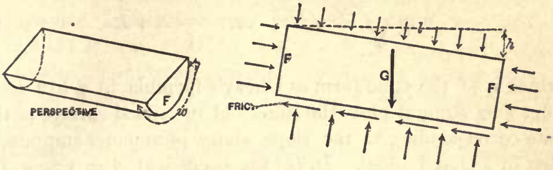

FIG. 610.

components parallel to the bed and surface are the *weight* $G = Fl\gamma$ of the prism (where $\gamma =$ heaviness of water) making an angle $= s$ (= slope) with a normal to the surface, and the *friction* between the water and the bed which is parallel to the surface. The amount of this friction for the prism in question may be expressed as in § 510, viz.:

$$P = fric. = fS\gamma\frac{v^2}{2g} = fwl\gamma\frac{v^2}{2g}, \quad . \quad . \quad . \quad (2)$$

in which $S = wl =$ rubbing surface (area) = wetted perimeter, w, × length (see § 538), and f an abstract number. Since the mass of water in Fig. 610 is supposed to be in relative equilibrium, we may apply to it the laws of motion of a rigid body, and since the motion is a *uniform translation* (§ 109) the components, parallel to the surface, of all the forces must balance.

$$\therefore\ G \sin s \text{ must} = P = fric.; \quad \therefore\ Fl\gamma\frac{h}{l} = fwl\gamma\frac{v^2}{2g};$$

whence

$$h = f\frac{wl}{F}\cdot\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)'$$

or

$$h = f\frac{l}{R}\cdot\frac{v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which $F \div w$ is called R, the *hydraulic mean depth*, or *hydraulic radius.* (3) is sometimes expressed by saying that the whole fall, or head, h, is (in uniform motion) absorbed in friction-head. Also, since the slope $s = h \div l$, we have

$$v = \sqrt{\frac{2g}{f}}\sqrt{Rs}; \quad \text{or,} \quad v = A\sqrt{Rs}, \quad . \cdot \quad . \quad . \quad (4)$$

which is of the same form as Chézy's formula in § 519 for a very long straight pipe (the slope s of the actual surface in this case corresponding to the slope along piezometer-summits in that of a closed pipe). In (4) the coefficient $A = \sqrt{2g \div f}$ is not, like f, an abstract number, but its numerical value depends on the system of units employed.

542a. Experiments on the Flow of Water in Open Channels.— Those of Darcy and Bazin, begun in 1855 and published in 1865 ("Recherches Hydrauliques"), were very carefully conducted with open conduits of a variety of shapes, sizes, slopes, and character of surface. In most of these a uniform flow was secured before the taking of measurements. The velocities ranged between from about 0.5 to 8 or 10 ft. per second, the hydraulic radii from 0.03 to 3.0 ft., with deliveries as high as 182 cub. ft. per second. For example, the following results were obtained in the canals of Marseilles and Craponne, the quantity A being for the foot and second. The sections were nearly all rectangular. See eq. (4) above.

No.	Q. (cub. ft. per sec.)	R. (ft.)	s. abs. numb.	v. (ft. per sec.)	A. (foot and sec.)	Character of the masonry surface.
1	182.73	1.504	.0037	10.26	137.1	Very smooth.
2	143.74	1.774	.00084	5.55	125.	Quite "
3	43.93	.708	.029	11.23	78.4	
4	43.93	.615	.060	13.93	72.5	Hammered stone.
5	43.93	.881	.0121	7.58	73.5	Rather rough.
6	43.93	.835	.014	8.36	77.3	
7	167.68	2.871	.00043	2.54	72.2	Mud and vegetation.

[In Experiment No. 7 the flow had not fully reached a state of permanency.]

Fteley and Stearns's experiments on the Sudbury conduit at Boston, Mass. (*Trans. A. S. C. E.*, '83), from 1878 to 1880, are also valuable. This open channel was of brick masonry with good mortar joints, and about 9 ft. wide; the depths of water ranging from 1.5 to 4.5 ft. With plaster of pure cement on the bed in one of the experiments the high value of $A = 153.6$ was reached (foot and second), with $v = 2.805$ ft. per second, $R = 2.111$ ft., $s = .0001580$, and $Q = 87.17$ cu. ft. per second.

Captain Cunningham, in his experiments on the Ganges Canal at Roorkee, India, in 1881, found A to range from 48 to 130 (foot and second).

Humphreys and Abbot's experiments on the Mississippi River and branches (see § 540), with values of $R =$ from 2 or 3 ft. to 72 ft., furnish values of $A =$ from 53 to 167 (foot and second).

542b. Kutter's Formula.—The experiments upon which Weisbach based his deductions for f, the coefficient of fluid friction, were scanty and on too small a scale to warrant general conclusions. That author considered that f depended only on the velocity, disregarding altogether the degree of roughness of the bed, and gave a table of values in accordance with that view, these values ranging from .0075 for 15 ft. per sec. to .0109 for 0.4 ft. per sec.; but in 1869 Messrs. Kutter and Ganguillet, having a much wider range of experimental data at command, including those of Darcy and Bazin, and those obtained on the Mississippi River, evolved a formula, known as *Kutter's Formula*, for the uniform motion of water in open channels, which is claimed to harmonize in a fairly satisfactory manner the chief results of the best experiments in that direction. They make the coefficient A in eq. (4) (or rather the factor $\frac{1}{\sqrt{f}}$ contained in A) a function of R, s, and also n an abstract number, or *coefficient of roughness*, depending on the nature of the surface of the bed and banks; viz.,

$$\left.\begin{array}{c} v\text{ in} \\ \text{ft.} \\ \text{per} \\ \text{sec.} \end{array}\right\} = \left[\frac{41.6 + \dfrac{1.811}{n} + \dfrac{.00281}{s}}{1 + \left(41.6 + \dfrac{.00281}{s}\right)\dfrac{n}{\sqrt{R(\text{in feet})}}}\right] \sqrt{R(\text{in ft.}) \times s}, \ldots (5)$$

which is *Kutter's Formula.*

That is, comparing (5) with (4), we have f a function of n, R, and s, as follows:

$$f = \left[\frac{1 + \left[41.6 + \frac{.00281}{s}\right]\frac{n}{\sqrt{R \text{ in ft.}}}}{5.184 + \frac{.2256}{n} + \frac{.00035}{s}}\right]^2 \quad . \quad . \quad . \quad (6)$$

From (6) it appears that f decreases with an increasing R, as has been also noted in the case of closed pipes (§ 517); that it increases with increasing roughness of surface; and that it is somewhat dependent on the slope. The makers of the formula give the following values for n.

Values of n. $n =$

.009 for well-planed timber bed;
.010 for plaster in pure cement;
.011 for plaster in cement with $\frac{1}{3}$ sand;
.012 for unplaned timber;
.013 for ashlar and brickwork; *
.015 for canvas lining on frames;
.017 for rubble;
.020 for canals in very firm gravel;
.025 for rivers and canals in perfect order and regimen, and perfectly free from stones and weeds;
.030 for rivers and canals in moderately good order and regimen, having stones and weeds occasionally;
.035 for rivers and canals in bad order and regimen, overgrown with vegetation and strewn with stones or detritus of any sort.

Kutter's Formula is claimed to apply to all kinds and sizes of watercourses, from large rivers to sewers and ditches; for *uniform motion.* If $\sqrt{R}$ is the unknown quantity, Kutter's Formula leads to a quadratic equation; if s the slope, to a cubic. Hence, to save computation, tables have been prepared, some of which will be found in vol. 28 of Van Nostrand's Magazine

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* For ordinary brick sewers Mr. R. F. Hartford claims that $n = .014$ gives good results. See Jour. Eng. Societies for '84–'85, p. 220.

(pp. 135 and 393) (sewers), and in Jackson's works on Hydraulics (rivers).

The following table will give the student an idea of the variation of the coefficient $A, = \sqrt{\frac{2g}{f}}$, of eq. (4), or large bracket of eq. (5), with different hydraulic radii, slopes, and values of n, according to Kutter's Formula; from $R = \frac{1}{2}$ ft., for a small ditch or sluice-way (or a wide and shallow stream), to $R = 15$ ft., for a river or canal of considerable size. Under each value of R are given two values of A; one for a slope of $s' = .001$, and the other for $s'' = .00005$. All these values of A imply the use of the foot and second.

These values of A have been scaled by the writer from a diagram given in Jackson's translation of Kutter's "*Hydraulic Tables*," and are therefore only approximate. The corresponding values of f, the coefficient of fluid friction, can be computed from $f = \frac{2g}{A^2}$.

n	$R = \frac{1}{2}$ ft.		$R = 1$ ft.		$R = 3$ ft.		$R = 6$ ft.		$R = 15$ ft.	
	for s'	for s''	for s'	for s''	for s'	for s''	for s'	for s''	for s'	for s''
0.010	133	114	149	137	174	174	187	196	199	222
0.015	83	68	96	87	118	118	128	137	138	158
0.020	58	49	70	63	87	87	98	106	110	126
0.025	45	38	54	48	70	70	80	85	90	106
0.030	36	31	43	40	58	58	66	72	77	90
0 035	28	25	38	34	50	50	58	64	68	81

The formula used in designing the New Aqueduct for New York City, in 1885, by Mr. Fteley, consulting engineer, was [see (4)]

$$v\,(ft.\ per\ sec.) = 142\sqrt{R(in\ ft.) \times s}, \quad . \quad . \quad . \quad (7)$$

whereas Kutter's Formula gives for the same case (a circular* section of 14 ft. diameter, and slope of 0.7 ft. to the mile), with $n = 0.013$,

$$v\,(ft.\ per\ sec.) = 140.7\sqrt{R(in\ ft.) \times s}. \quad . \quad . \quad . \quad (8)$$

---

* The aqueduct has this circular form for a small portion, only, of its length; a "horseshoe" section of very nearly the same flowing capacity being given to the greater portion of the remainder.

To quote from a letter of Mr. I. A. Shaler of the Aqueduct Corps of Engineers, "Mr. Fteley states that the cleanliness of the conduit (Sudbury) had much to do in affecting the flow. He found the flow to be increased by 7 or 8 per cent in a portion which had been washed with a thin wash of Portland cement."

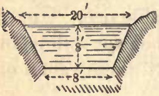

FIG. 611.

EXAMPLE 1.—A canal 1000 ft. long of the trapezoidal section in Fig. 611 is required to deliver 300 cubic ft. of water per second with the water 8 ft. deep at all sections (i.e., with uniform motion), the slope of the bank being such that for a depth of 8 ft. the width of the water surface (or length of air-profile) will be 20 ft.; and the coefficient for roughness being $n = .020$. What is the necessary slope to be given to the bed (slope of bed = that of surface, here) (ft., lb., sec.)?

The mean velocity

$$v = Q \div F = 300 \div \tfrac{1}{2}(20 + 8)\,8 = 2.67 \text{ ft. per sec.}$$

[So that the surface velocity of mid-channel in any section would probably be $(c_{0\,\text{max.}}) = v \div 0.83 = 3.21$ ft. per sec. (eq. (2), § 540).]

The wetted perimeter

$$w = 8 + 2\sqrt{8^2 + 6^2} = 28 \text{ ft.},$$

and therefore the mean hydraulic depth

$$= R = F \div w = 112 \div 28 = 4 \text{ ft.}$$

To obtain a first approximation for the slope, we may use the value $f = .00795$ given by Weisbach for a velocity of 2.67 ft. per sec., and obtain, from (3),

$$h = \frac{.00795 \times 1000 \times 28\,(2.67)^2}{112 \times 2 \times 32.2} = 0.221 \text{ ft.};$$

i.e., $s = h \div l = .000221$.

With this value for the slope and $R = 4$ ft. (see above), we then have, from eq. (6) (putting $n = .020$),

$$f = \left[\frac{1 + \left(41.6 + \frac{.00281}{.000221}\right)\frac{.020}{\sqrt{4 \text{ ft.}}}}{5.184 + \frac{.2256}{.020} + \frac{.00035}{.000221}}\right]^2 = .0071,$$

with which value of f we now obtain

$$h = 0.200 \text{ feet}\,; \quad \text{i.e.,} \quad \text{slope} = s = .00020.$$

Example 2.—If the bed of a creek falls 20 inches every 1500 ft. of length, what volume of water must be flowing to maintain a uniform mean depth of 4½ ft., the corresponding surface-width being 40 ft., and wetted perimeter 46 ft.? The bed is "in moderately good order and regimen;" use Kutter's Formula, putting $n = 0.030$ (ft. and sec.).

First we have

$$\sqrt{Rs} = \sqrt{(40 \times 4\tfrac{1}{2}) \div \left(46 \times \frac{1500}{\frac{20}{12}}\right)} = .066,$$

while $\sqrt{R}$ (ft.) $=1.98$, and the slope $= s = \frac{20}{12} \div 1500 = .00111$; hence

$$v = \frac{\left[41.6 + \frac{1.811}{.030} + \frac{.00281}{.00111}\right] \times 0.066}{1 + \left[41.6 + \frac{.00281}{.00111}\right]\frac{0.030}{1.98}} = \frac{104.43 \times .066}{1.6685},$$

or

$$v = 4.13 \text{ ft. per sec.}$$

Hence, also,

$$Q = Fv = 40 \times 4\tfrac{1}{2} \times 4.13 = 743.4 \text{ cub. ft. per sec.}$$

[N.B. Weisbach works this same example by eq. (3) with a value of f taken from his own table, his result being $v = 6.1$

ft. per sec., which would probably be attained in practice only by making the bed and banks smoother than as given.]

EXAMPLE 3.—The desired transverse water-section of a canal is given in Fig. 612. The slope is to be 3 ft. in 1600; i.e., $s = 3 \div 1600$; or, for $l = 1600$ ft., $h = 3$ ft. What must be the velocity (mean) of each section, for a *uniform motion*; the corresponding volume delivered per sec., $Q, = Fv, = ?$; assuming that the character of the surface warrants the value $n = .030$?

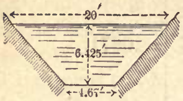

FIG. 612.

Knowing the slope $s, = 3 \div 1600$; and the hydraulic radius $R, = F \div w, = 79.28$ sq. ft. $\div$ 24.67 ft., $= 3.215$ feet; with $n = .030$ we substitute directly in eq. (5), obtaining $v = 4.67$ ft. per sec.; whence $Q = Fv = 370$ cub. ft. per sec.

543. Hydraulic Mean Depth for a Minimum Frictional Resistance.—We note, from eq. (3), § 542, that if an open channel of given length l and sectional area F is to deliver a given volume, Q, per time-unit with uniform motion, so that the common mean velocity v of all sections ($= Q \div F$) is also a given quantity, the necessary fall $= h$, or slope $s = h \div l$, is seen to be inversely proportional to R, the hydraulic mean depth of the section, $= (F \div w), =$ sectional area $\div$ wetted perimeter.

For h to be as small as possible, we may design the form of transverse section, so as to make R as large as possible; i.e., to make the wetted perimeter a minimum for a given F; for in this way a minimum of frictional contact, or area of rubbing surface, is obtained for a prism of water of given sectional area F and given length l.

In a closed pipe running full the wetted perimeter is the whole perimeter; and if the given sectional area is shaped in the form of a *circle*, the wetted perimeter, $= w$, is a minimum (and R a maximum). If the full pipe must have a polygonal shape of n sides, then the *regular* polygon of n sides will provide a minimum w.

Whence it follows that if the pipe or channel is running half full, and thus becomes an *open channel*, the semicircle, of all curvilinear water profiles, gives a minimum w. Also, of all trapezoidal profiles with banks at 60° with the horizontal the half of a regular hexagon gives a minimum w. Among all rectangular sections the half square gives a minimum w; and of all half octagons the half of a regular octagon gives a minimum w (and max. R) for a given F. See Fig. 613 for all these.

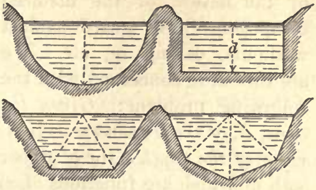

FIG. 613.

The egg-shaped outline, Fig. 614, small end down, is frequently given to sewers in which it is important that the different velocities of the water at different stages (depths) of flow (depending on the volume of liquid passing per unit-time) should not vary widely from each other. The lower portion ABC, providing for the lowest stage of flow AB, is nearly semicircular, and thus induces a velocity of flow (the slope being constant at all stages) which does not differ extremely from that occurring when the water flows at its highest stage DE, although this latter velocity is the greater; the reason being that ABC from its advantageous form has a hydraulic radius, R, larger in proportion to its sectional area, F, than DCE.

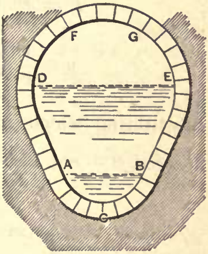

FIG. 614.

That is, $F \div w$ for ABC is more nearly equal to $F \div w$ for DEC than if DEC were a semicircle, and the velocity at the lowest stage may still be sufficiently great to prevent the deposit of sediment. See § 575.

544. Trapezoid of Fixed Side-slope.—For large artificial watercourses and canals the trapezoid, or three-sided water-profile (symmetrical), is customary, and the inclination of the bank, or angle θ with the horizontal, Fig. 615, is often determined by the nature of the material composing it, to guard against washouts, caving in, etc. We are therefore concerned with the following problem: *Given the area, F, of the transverse section, and the angle θ, required the value of the depth x* (or of upper width z, or of lower width y, both of which are functions of x) *to make the hydraulic mean depth, $R = F \div w$, a maximum*, or *$w \div F$ a minimum.* F is constant.

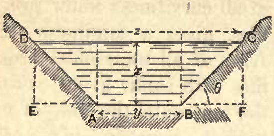

FIG. 615.

From the figure we have

$$w = \overline{AB} + 2\overline{BC} = y + 2x \operatorname{cosec.} \theta, \quad . \quad . \quad . \quad (1)$$

and

$$F = yx + x^2 \cot. \theta;$$

whence

$$y = \frac{1}{x} . (F - x^2 \cot. \theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

substituting which in (1) and dividing by F, noting that $2 \operatorname{cosec.} \theta - \cot. \theta = \frac{2 - \cos\theta}{\sin\theta}$, we have

$$\frac{w}{F} = \frac{1}{R} = \frac{1}{x} + \frac{2 - \cos\theta}{F \sin\theta} . x. \quad . \quad . \quad . \quad . \quad (3)$$

For a minimum w we put

$$\frac{d\left(\frac{w}{F}\right)}{dx} = 0; \quad \text{i.e.,} \quad -\frac{1}{x^2} + \frac{2 - \cos\theta}{F \sin\theta} = 0;$$

$$\therefore \; x \, (\text{for max. or min. } w) = \pm \sqrt{\frac{F \sin\theta}{2 - \cos\theta}}.$$

The $+$ sign makes the second derivative positive, and hence for a min. w or max. R we have

$$x \, (\text{call it } x') = x' = \frac{\sqrt{F \sin\theta}}{\sqrt{2 - \cos\theta}}, \quad . \quad . \quad . \quad (4)$$

while the corresponding values for the other dimensions are

$$y' = \frac{F}{x'} - x' \cot. \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$z' = y' + 2x' \cot. \theta = \frac{F}{x'} + x' \cot. \theta. \quad . \quad . \quad . \quad (6)$$

For the corresponding hydraulic mean depth R' [see (3)], i.e., the max. R, we have

$$\frac{1}{R'} = \frac{1}{x'} + \frac{2 - \cos \theta}{F \sin \theta} x' = \frac{2}{x'}; \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore \ R' = \tfrac{1}{2} x' = \frac{1}{2} \sqrt{\frac{F \sin \theta}{2 - \cos \theta}}. \quad . \quad . \quad . \quad . \quad (8)$$

Equations (4), (5), . . . (8) hold good, then, for the trapezoidal section of least frictional resistance for a given angle θ.

Problem.—Required the dimensions of the trapezoidal section of minimum frictional resistance for $\theta = 45°$, which with $h = 6$ inches fall in every 1200 feet $(= l)$ is required to deliver $Q = 360$ cub. ft. of water per minute with *uniform motion.*

Here we have given, with uniform motion, h, l, and Q, with the requirement that the section shall be trapezoidal, with $\theta = 45°$, and of minimum frictional resistance. The following equations are available:

Eq. of continuity . . . $Q = Fv$, ... (1′)

Eq. (8) preceding, for condition of least resistance } . $R' = \frac{1}{2} \sqrt{\frac{\sin \theta}{2 - \cos \theta}} \sqrt{F}$. . (2′)

From eq. (3), § 542, } for uniform motion, $h = f \frac{l}{R} \cdot \frac{v^2}{2g}$. . . . . . (3′)

There are three unknown quantities, v, F, and R'. Solve

(1′) for v; solve (2′) for R'; substitute their values in (3′); whence

$$\frac{h\sqrt{\sin\ \theta}}{\sqrt{2-\cos\theta}}=\frac{f}{2g}\cdot\frac{2l}{\sqrt{F}}\cdot\frac{Q^2}{F^2};\ \therefore\ F=\left[\frac{2fl\sqrt{2-\cos\theta}\,Q^2}{2gh\sqrt{\sin\theta}}\right]^{\frac{2}{5}}.\quad (4')$$

Since f cannot be exactly computed in advance, for want of knowing the value of R, we calculate it approximately [eq. (6), § 542b] for an assumed value of R, insert it in the above equation (4′), and thus find an approximate value of F; and then, from (8), a corresponding value of R, from which a new value of f can be computed. Thus after one or two trials a satisfactory adjustment of dimensions can be secured.

545. Variable Motion.—If a steady flow of water of a delivery $Q, = Fv, =$ constant, takes place in a straight open channel the slope of whose bed has not the proper value to maintain a "*uniform motion*," then "*variable motion*" ensues (the flow is still steady, however); i.e., although the mean velocity in any one transverse section remains fixed (with lapse of time), this velocity has different values for different sections; but as the eq. of continuity,

$$Q=Fv=F_1v_1=F_2v_2,\ \text{etc.},$$

still holds (since the flow is steady), the different sections have different areas. If, Fig. 616, a stream of water flows down an inclined trough without friction, the relation between the velocities v_0 and v_1 at any two sections 0 and 1 will be the same as for a material point sliding down a guide without friction (see § 79, latter part), viz.:

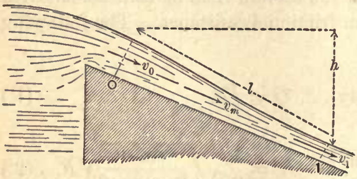

FIG. 616.

$$\frac{v_1^2}{2g}=\frac{v_0^2}{2g}+h,\ \ldots\ldots\ldots\ (1)$$

an equation of heads (really a case of Bernoulli's Theorem, § 492). But, considering friction on the bed, we must subtract the *mean friction-head* $f\frac{l}{R}\cdot\frac{v_m^2}{2g}$ [see eqs. (3) and (3′), § 542] lost between 0 and 1; this friction-head may also be written thus: $f\frac{lw_m}{F_m}\frac{v_m^2}{2g}$; and therefore eq. (1) becomes

$$\frac{v_1^2}{2g} = \frac{v_0^2}{2g} + h - \frac{fw_m}{F_m}l\cdot\frac{v_m^2}{2g}, \quad . \quad . \quad . \quad . \quad (2)$$

which is the formula for *variable motion;* and in it l is the length of the section considered, which should be taken short enough to consider the surface straight between the end-sections, and the latter should differ but slightly in area. The subscript m may be taken as referring to the section midway between the ends, so that $v_m^2 = \frac{1}{2}(v_0^2 \wedge + v_1^2)$. The wetted perimeter $w_m = \frac{1}{2}(w_0 + w_1)$, and $F_m = \frac{1}{2}(F_0 + F_1)$. Hence eq. (2) becomes $2v_0v$

$$+v_0v_1$$

$$h = \frac{v_1^2}{2g} - \frac{v_0^2}{2g} + \frac{\frac{1}{2}f(w_0 + w_1)l}{F_0 + F_1}\cdot\frac{v_0^2 \wedge + v_1^2}{2g}; \quad . \quad . \quad (3)$$

and again, by putting $v_0 = Q \div F_0$, $v_1 = Q \div F_1$, we may write

$$h = \left[\frac{1}{F_1^2} - \frac{1}{F_0^2} + \frac{1}{\overset{4}{2}}\cdot\frac{fl(w_0 + w_1)}{F_0 + F_1}\left(\frac{1}{F_0^2}\overset{+\frac{2}{F_0F_1}}{\wedge +}\frac{1}{F_1^2}\right)\right]\frac{Q^2}{2g}; \; . \; (4)$$

whence

$$Q = \frac{\sqrt{2gh}}{\sqrt{\frac{1}{F_1^2} - \frac{1}{F_0^2} + \frac{1}{2}\cdot\frac{fl(w_0 + w_1)}{F_0 + F_1}\cdot\left[\frac{1}{F_0^2} + \frac{1}{F_1^2}\right]}}. \quad (5)$$

From eq. (4), having given the desired shapes, areas, etc., of the end-sections and the volume of water, Q, to be carried per unit of time, we may compute the necessary fall, h, of the surface in length $= l$; while from eq. (5), having observed in an actual water-course the values of the sectional areas F_0 and F_1; the wetted perimeters w_0 and w_1, the length, $= l$, of the por tion considered, we may calculate Q and thus *gauge* the stream approximately, without making any velocity measurements.

As to the value of f, we compute it from eq. (6), § 542b, using for R a mean between the values of the hydraulic radii of the end-sections.

546. Bends in an Open Channel.—According to Humphreys and Abbot's researches on the Mississippi River the loss of head due to a bend may be put

$$h_r = \frac{v^2}{536} \frac{6\delta}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which v must be in *ft. per sec.*, and δ, the angle ABC, Fig. 617, must be in π-measure, i.e. in radians. The section F must be greater than 100 sq. ft., and the slope s less than .0008. v is the mean velocity of the water. Hence if a bend occurred in a portion of a stream of length l, eq. (3) of § 542 becomes

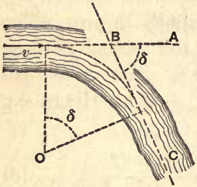

FIG. 617.

$$h = \frac{fl}{R} \frac{v^2}{2g} + \frac{6}{536} \frac{v^2\delta}{\pi} \,.\,.\, \text{[ft. and sec.]}, \,.\, (2)$$

while eq. (2) of § 545 for variable motion would then become

$$\frac{v_1^2}{2g} = \frac{v_0^2}{2g} + h - \frac{f w_m l}{F_m} \frac{v_m^2}{2g} - \frac{6}{536} \frac{v^2\delta}{\pi} \,.\,.\, \text{(ft. and sec.)}. \,.\, (3)$$

(v and δ as above.) (For "radian" see p. 544.)

547. Equations for Variable Motion, introducing the Depths.—Fig. 618. The slope of the bed being $\sin \alpha$ (or simply α, π meas.), while that of the surface is different, viz.,

$$\sin \beta = s = h \div l,$$

we may write

$$h = d_0 + l \sin \alpha - d_1,$$

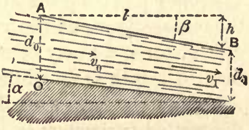

FIG. 618.

in which d_0 and d_1 are the depths at the end-sections of the portion considered (steady flow with variable motion). With these substitutions in eq. (4), § 545, we have, solving for l,

$$l = \frac{d_0 - d_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{\frac{fw_m}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - \sin\alpha}. \quad . \quad . \quad . \quad (1)$$

From which, knowing the slope of the bed and the shape and size of the end-sections, also the discharge Q, we may compute the length or distance, l, between two sections whose depths differ by an assigned amount $(d_0 - d_1)$. But we cannot compute the change of depth for an assigned length l from (6). However, if the *width* b of the stream is *constant*, and the same at all depths; i.e., if all sections are rectangles having a common width; eq. (6) may be much simplified by introducing some approximations, as follows: We may put

$$\left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \frac{F_0^2 - F_1^2}{F_1^2 F_0^2}\cdot\frac{Q^2}{2g} = \frac{(F_0 - F_1)(F_0 + F_1)}{F_1^2}\cdot\frac{v_0^2}{2g}$$

$$= \frac{(d_0 - d_1)(d_0 + d_1)}{d_1^2}\cdot\frac{v_0^2}{2g}, \text{ which approx.} = \frac{2(d_0 - d_1)}{d_0}\cdot\frac{v_0^2}{2g}.$$

and, similarly,

$$\frac{w_m}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} = \frac{w_m (F_0^2 + F_1^2}{(F_0 + F_1) F_1^2}\cdot\frac{v_0^2}{2g}$$

$$\text{which approx.} = \frac{w_m}{d_0 b}\,\frac{v_0^2}{2g}.$$

Hence by substitution in eq. (6) we have

$$l = \frac{(d_0 - d_1)\left[1 - \frac{2}{d_0}\cdot\frac{v_0^2}{2g}\right]}{\frac{fw_m}{d_0 b}\,\frac{v_0^2}{2g} - \sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

547a. Backwater.—Let us suppose that a steady flow has been proceeding with uniform motion (i.e., the surface parallel to the bed) in an open channel of indefinite extent, and that a vertical wall is now set up across the stream. The water rises and flows over the edge of the wall, or weir, and after a time a steady flow is again established. The depth, y_0, of the water close to the weir on the up-stream side is greater than d_0, the original depth. We now have "variable motion" above the weir, and at any distance x up-stream from the weir the new depth y is greater than d_0. This increase of depth is called backwater, and, though decreasing up-stream, may be perceptible several miles above the weir. Let s be the slope of the original uniform motion (and also of present bed), and v the velocity of the original uniform motion, and let $k = \frac{v^2}{2g}$. Then, if the section of the stream is a shallow rectangle of constant width, we have the following relation (Rankine):

$$x = \frac{1}{s}\left[y_0 - y + (d_0 - 2k)(\phi - \phi_0)\right], \quad . \quad . \quad . \quad (1)$$

where ϕ is a function of $\frac{y}{d_0}$, as per following table:

For $\frac{y}{d_0} = 1.0$	1.10	1.20	1.30	1.40	1.50	1.60	1.70
$\phi = \infty$	.680	.480	.376	.304	.255	.218	.189
For $\frac{y}{d_0} = 1.80$	1.90	2.00	2.20	2.40	2.60	2.80	3.0
$\phi = .166$	.147	.132	.107	.089	.076	.065	.056

ϕ_0 is found from $\frac{y_0}{d_0}$, precisely as ϕ from $\frac{y}{d_0}$, by use of the table. With this table and eq. (1), therefore, we can find x, the distance ("amplitude of backwater") from the weir of the point where any assigned depth y (or "height of backwater," $y - d_0$) will be found.

For example, Prof. Bowser cites the case from D'Aubuisson's Hydraulics of the river Weser in Germany, where the erection of a weir increased the depth at the weir from 2.5 ft. to 10 ft., the flow having been originally "uniform" for 10 miles. Three miles above the dam the *increase* $(y - d_0)$ of depth was 1.25 ft., and even at four miles it was 0.75 ft.

CHAPTER VIII.

DYNAMICS OF GASEOUS FLUIDS.

548. Steady Flow of a Gas.—[N.B. The student should now review § 492 up to eq. (5).] The differential equation from which Bernoulli's Theorem was derived for any liquid, *without friction*, was [eq. (5), § 492]

$$\frac{1}{g}\,vdv + dz + \frac{1}{\gamma}\,dp = 0, \quad . \quad . \quad . \quad . \quad . \quad (A)$$

and is equally applicable to the steady flow of a gaseous fluid, but with this difference in subsequent work, that the heaviness, γ (§ 7), of the gas passing different sections of the pipe or stream-line is, or may be, different (though always the same at a given point or section, since the flow is steady). For the present we neglect friction and consider the flow from a large receiver, where the great body of the gas is practically at rest, through an orifice in a thin plate, or a short nozzle with a rounded entrance.

In the steady flow of a gas, since γ is different at different points, the *equation of continuity* takes the form

$$\textit{Flow of weight per time-unit} = F_1v_1\gamma_1 = F_2v_2\gamma_2 = \text{etc.}\,; \quad . \quad (a)$$

i.e., the *weight* of gas passing any section, of area F, per unit of time, is the same as for any other section, or $Fv\gamma =$ constant, γ being the heaviness at the section, and v the velocity.

549. Flow through an Orifice—Remarks.—In Fig. 619 we have a large rigid receiver containing gas at some tension, p_n, higher than that, p_m, of the (still) outside air (or gas), and at some absolute temperature T_n, and of some heaviness γ_n; that is, in a *state n*. The small orifice of area F being opened, the gas begins to escape, and if the receiver is very large, or if the supply is continually kept up (by a blowing-engine, e.g.), after

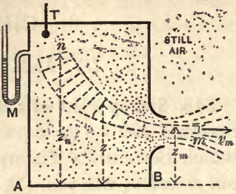

FIG. 619.

a very short time the flow becomes steady. Let nm represent any stream-line (§ 495) of the flow. According to the ideal subdivision of this stream-line into laminæ of equal mass or weight (not equal volume, necessarily) in establishing eq. (A) for any one lamina, each lamina in the lapse of time dt moves into the position just vacated by the lamina next in front, and *assumes precisely the same velocity, pressure, and volume (and therefore heaviness) as that front one had at the beginning of the* dt. In its progress toward the orifice it expands in volume, its tension diminishes, while its velocity, insensible at n, is gradually accelerated on account of the pressure from behind always being greater than that in front, until at m, in the "*throat*" of the jet, the velocity has become v_m, the pressure (i.e., tension) has fallen to a value p_m, and the heaviness has changed to γ_m. The temperature T_m (absolute) is less than T_n, since the expansion has been rapid, *and does not depend on the temperature of the outside air or gas into which efflux takes place*, though, of course, after the effluent gas is once free from the orifice it may change its temperature in time.

We assume the pressure p_m (in throat of jet) to be equal to that of the outside medium (as was done with flow of water), so long as that outside tension is greater than .527 p_n; but if it is less than .527 p_n and is even zero (a vacuum), experiment seems to show that p_m remains equal to 0.527 of the interior tension p_n: probably on account of the expansion of the effluent gas beyond the throat, Fig. 620, so that although the tension in the outer edge, at a, of the jet is equal to that of the outside medium, the tension at m is greater because of the centripetal and centrifugal forces developed in the curved filaments between a and m. (See § 553.)

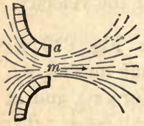

FIG. 620.

550. Flow through an Orifice; Heaviness assumed Constant during Flow. The Water Formula.—If the inner tension p_n exceeds the outer, p_m, but slightly, we may assume that, like water, the gas remains of the same heaviness during flow. Then, for the simultaneous advance made by all the laminæ of a stream-line, Fig. 619, in the time dt, we may conceive an equation like eq. (A) written out for each lamina between n and m, and corresponding terms added; i.e.,

$$(\textit{For orifices}) \quad \ldots \quad \frac{1}{g}\int_n^m v\,dv + \int_n^m dz + \int_n^m \frac{dp}{\gamma} = 0. \quad . \; (B)$$

In general, γ is different in the different laminæ, but in the present case it is assumed to be the same in all; hence, with m as datum level and h = vertical distance from n to m, we have, from eq. (B),

$$\frac{v_m^2}{2g} - \frac{v_n^2}{2g} + 0 - h + \frac{p_m}{\gamma} - \frac{p_n}{\gamma} = 0. \quad . \quad . \quad . \quad (1)$$

But we may put $v_n = 0$; while h, even if several feet, is small compared with $\frac{p_n}{\gamma} - \frac{p_m}{\gamma}$. E.g., with $p_m = 15$ lbs. per sq. in. and $p_n = 16$ lbs. per sq. in., we have for atmospheric air at freezing temperature

$$\frac{p_n}{\gamma} - \frac{p_m}{\gamma} = 1638 \text{ feet.}$$

Hence, putting $v_n = 0$ and $h = 0$ in eq. (1), we have

$$\frac{v_m^2}{2g} = \frac{p_n - p_m}{\gamma_n} \quad \ldots \quad \left\{ \begin{array}{l} \textit{Water formula ; for small} \\ \textit{difference of pressures, only.} \end{array} \right\} \quad \ldots \quad (2)$$

The interior absolute temperature T_n being known, the γ_n (interior heaviness) may be obtained from $\gamma_n = p_n \gamma_0 T_0 \div T_n p_0$ (§ 472), and the volume of flow per unit of time then obtained (first solving (2) for v_m) is

$$Q_m = F_m v_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where F_m is the sectional area of the jet at m. If the mouthpiece or orifice has well-rounded interior edges, as in Fig. 541, its sectional area F may be taken as the area F_m. But if it is an orifice in "thin plate," putting the coefficient of contraction $= C = 0.60$, we have

$$F_m = CF = 0.60\ F; \quad \text{and} \quad Q_m = 0.60\ Fv_m. \quad . \quad (4)$$

This volume, Q_m, is that occupied by the flow per time-unit when in *state* m, and we have assumed that $\gamma_m = \gamma_n$; hence the *weight of flow* per time-unit is

$$G = Q_m\gamma_m = F_m v_m \gamma_m = F_m v_m \gamma_n. \quad . \quad . \quad . \quad (5)$$

Example.—In the testing of a blowing-engine it is found capable of maintaining a pressure of 18 lbs. per sq. inch in a large receiver, from whose side a blast is steadily escaping through a "thin plate" orifice (circular) having an area $F = 4$ sq. inches. The interior temperature is 30° Cent. and the outside tension 15 lbs. per sq. in.

Required the discharge of air per second, both volume and weight. The data are: $p_n = 18$ lbs. per sq. in., $T_n = 303°$ Abs. Cent., $F = 4$ sq. inches, and $p_m = 15$ lbs. per sq. in. Use ft.-lb.-sec. system.

First, the heaviness in the receiver is

$$\gamma_n = \frac{p_n}{p_0} \cdot \frac{T_0}{T_n}\gamma_0 = \frac{18}{14.7} \cdot \frac{273}{303} \times .0807 = .089 \text{ lbs. per cub. ft.}$$

Then, from eq. (2),

$$v_m = \sqrt{2g\frac{p_n - p_m}{\gamma_n}} = \sqrt{\frac{2 \times 32.2[144 \times 18 - 144 \times 15]}{0.089}} = \left\{ \begin{matrix} 555.3 \\ \text{feet} \\ \text{per sec.} \end{matrix} \right.$$

(97 per cent of this would be more correct on account of friction.)

$$\therefore \ Q_m = F_m v_m = .6Fv_m = \tfrac{6}{10} \cdot \tfrac{4}{144} \times 555.3 = 9.24 \text{ cub. ft. per sec.}$$

at a tension of 15 lbs. per sq. in., and of heaviness (by hypothesis) $= .089$ lbs. per cub. ft. Hence weight

$$= G = 9.24 \times .089 = .82 \text{ lbs. per sec.}$$

The theoretical *power* of the air-compressor or blowing-engine to maintain this steady flow can be computed as in Example 3, § 483.

551. Flow through an Orifice on the Basis of Mariotte's Law; or Isothermal Efflux.—Since in reality the gas expands during flow through an orifice, and hence changes its heaviness (Fig. 619), we approximate more nearly to the truth in assuming this change of density to follow Mariotte's law, i.e., that the *heaviness varies directly as the pressure*, and thus imply that the *temperature remains unchanged during the flow.* We again integrate the terms of eq. (B), but take care to note that, *now*, γ is variable (i.e., different in different laminæ at the same instant), and hence express it in terms of the variable p (from eq. (2), § 475), thus:

$$\gamma = (\gamma_n \div p_n)p.$$

Therefore the term $\int_n^m \frac{dp}{\gamma}$ of eq. (B) becomes

$$\frac{p_n}{\gamma_n}\int_n^m \frac{dp}{p} = -\frac{p_n}{\gamma_n}\log._e\frac{p_n}{p_m}, \quad . \quad . \quad . \quad . \quad (1)$$

and, integrating all the terms of eq. (B), neglecting h, and calling v_n zero, we have

$$\frac{v_m^2}{2g} = \frac{p_n}{\gamma_n}\log._e\frac{p_n}{p_m} \; . \; . \; . \left\{\begin{array}{l}\textit{efflux by Mariotte's}\\ \textit{Law through orifice}\end{array}\right\} . \; . \; . \; (2)$$

As before, $\gamma_n = \frac{T_0}{T_n}\cdot\frac{p_n}{p_0}\gamma_0$, and the flow of volume per time-unit at m is

$$Q_m = F_m v_m ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while if the orifice is in thin plate, F_m may be put $= .60\ F$, and the

$$\textit{weight of the flow per time-unit} = G = F_m v_m \gamma_m . \quad . \quad (4)$$

If the mouth-piece is rounded, $F_m = F =$ area of exit orifice of mouth-piece.

EXAMPLE.—Applying eq. (2) to the data of the example in § 550, where γ_n was found to be .089 lbs. per cub. ft., we have [ft., lb., sec.]

$$v_m = \sqrt{2g\frac{p_n}{\gamma_n}\log._e\frac{p_n}{p_m}}$$

$$= \sqrt{2 \times 32.2 \times \frac{18 \times 144}{.089} \times 2.3025 \times \log._{10}\left[\frac{18}{15}\right]} = 584.7 \text{ ft. p. sec.}$$

$$\therefore\ Q_m = F_m v_m = 0.60 \times \tfrac{4}{144} \times 584.7 = 9.745 \text{ cub. ft. per sec.}$$

Since the heaviness at m is, from Mariotte's law,

$$\gamma_m = \frac{p_m}{p_n}\gamma_n = \tfrac{15}{18} \text{ of } .089, \quad \text{i.e.,} \quad \gamma_m = .0741 \text{ lbs. per cub. ft.,}$$

hence the weight of the discharge is

$$G = Q_m\gamma_m = 9.745 \times .0741 = 0.722 \text{ lbs. per sec.,}$$

or about 12 per cent less than that given by the "water formula." If the difference between the inner and outer tensions had been less, the discrepancy between the results of the two methods would not have been so marked.

552. Adiabatic Efflux from an Orifice.—It is most logical to assume that the expansion of the gas approaching the orifice, being rapid, is *adiabatic* (§ 478). Hence (especially when the difference between the inner and outer tensions is considerable) it is more accurate to assume γ as varying according to Poisson's Law, eq. (1), § 478; i.e., $\gamma = [\gamma_n \div p_n^{\frac{2}{3}}]p^{\frac{2}{3}}$, in integrating eq. ($B$). Then the term

$$\int_n^m \frac{dp}{\gamma} \text{ will } = \frac{p_n^{\frac{2}{3}}}{\gamma_n}\int_n^m p^{-\frac{2}{3}}\,dp = \frac{3p_n^{\frac{2}{3}}}{\gamma_n}[p_m^{\frac{1}{3}} - p_n^{\frac{1}{3}}]$$

$$= -\frac{3p_n}{\gamma_n}\left[1 - \left(\frac{p_m}{p_n}\right)^{\frac{1}{3}}\right];$$

and eq. (B), neglecting h as before, and with $v_n = 0$, becomes (See Fig. 619)

$$\frac{v_m^2}{2g} = \frac{3p_n}{\gamma_n}\left[1 - \left(\frac{p_m}{p_n}\right)^{\frac{1}{3}}\right] . (\textit{Adiabatic flow; orifice.}) . (1)$$

Having observed p_n and T_n in the reservoir, we compute $\gamma_n = \frac{p_n \gamma_0 T_0}{T_n p_0}$ (from § 472). The gas at m, just leaving the orifice, having expanded adiabatically from the *state n* to the *state m*, has cooled to a temperature T_m (absolute) found thus (§ 478),

$$T_m = T_n \left(\frac{p_m}{p_n}\right)^{\frac{1}{3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and is of a heaviness

$$\nu_m = \gamma_n \left(\frac{p_m}{p_n}\right)^{\frac{2}{3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the flow per second occupies a volume (immediately on exit)

$$Q_m = F_m v_m , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and weighs

$$G = F_m v_m \gamma_m . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

EXAMPLE 1.—Let the interior conditions in the large reservoir of Fig. 619 be as follows (*state n*): $p_n = 22\frac{1}{2}$ lbs. per sq. in., and $T_n = 294°$ Abs. Cent. (i.e., 21° Cent.); while externally the tension is 15 lbs. per sq. inch, which may be taken as being $= p_m =$ tension at m, the throat of jet. The opening is a circular orifice in "thin plate" and of one inch diameter. Required the weight of the discharge per second [ft., lb., sec.; $g = 32.2$].

First, $\gamma_n = \frac{22.5 \times 144}{15 \times 144} \cdot \frac{273}{294} \times .0807 = 0.114$ lbs. per cub. ft.

Then, from (1),

$$v_m = \sqrt{2g\frac{3p_n}{\gamma_n}\left[1 - \left(\frac{p_m}{p_n}\right)^{\frac{1}{3}}\right]}$$

$$= \sqrt{\frac{2 \times 32.2 \times 3 \times 22.5 \times 144}{0.114}\left[1 - \sqrt[3]{\tfrac{2}{3}}\right]} = 844 \text{ ft. per sec.}$$

Now $F = \frac{1}{4}\pi(\frac{1}{12})^2 = .00546$ sq. ft.

$\therefore\ Q_m = CFv_m = .60\,Fv_m = 0.60 \times .00546 \times 844 = 2.765$ cub. ft. per sec., at a temperature of

$$T_m = 294\sqrt[3]{\tfrac{2}{3}} = 257° \text{ Abs. Cent.} = -16° \text{ Cent.},$$

and of a heaviness

$$\gamma_m = 0.114\sqrt[3]{(\tfrac{2}{3})^2} = 0.085 \text{ lbs. per cub. ft.},$$

so that the weight of flow per sec.

$$= G = Q_m\gamma_m = 2.765 \times .085 = .235 \text{ lbs. per sec.}$$

EXAMPLE 2.—Let us treat the example already solved by the two preceding approximate methods (§§ 550 and 551) by the present more accurate equation of adiabatic flow, eq. (1).

The data were (Fig. 619):

$p_n = 18$ lbs. per sq. in.; $T_n = 303°$ Abs. Cent.;
$p_m = 15$ " " " ; and $F = 4$ sq. inches

[F being the area of orifice]. γ_n was found $= .089$ lbs. per cub. ft. in § 550; hence, from eq. (1),

$$v_m = \sqrt{\frac{2 \times 32.2 \times 3 \times 18 \times 144}{.089}\left[1 - \sqrt[3]{\tfrac{5}{6}}\right]} = 576.2 \text{ ft. per sec.}$$

From (4),

$$Q_m = F_m v_m = .6Fv_m = .6 \times \tfrac{4}{144} \times 576.2 = 9.603 \text{ cub. ft. per sec.;}$$

and since at m it is of a heaviness

$$\gamma_m = .089\sqrt[3]{(\tfrac{15}{18})^2} = .0788 \text{ lbs. per cub. ft.},$$

we have weight of flow per sec.

$$= G = Q_m\gamma_m = 9.603 \times .0788 = 0.756 \text{ lbs. per sec.}$$

Comparing the three methods for this problem, we see that

By the "*water formula*," . . . $G = 0.82$ lbs. per sec.
" *isothermal formula*, . . $G = 0.722$ " "
" *adiabatic formula*, . . $G = 0.756$ " "

553. Practical Notes. Theoretical Maximum Flow of Weight. —If in the equations of § 552 we write for brevity $p_m \div p_n = x$ we derive, by substitution from (1) and (3) in (5),

$$\left.\begin{matrix}\text{Weight of flow}\\ \text{per unit of time}\end{matrix}\right\} = G = Q_m \gamma_m = F_m \sqrt{6 g p_n \gamma_n}\,[1 - x^{\frac{1}{3}}]^{\frac{1}{2}} x^{\frac{2}{3}}. \quad . \quad (1)$$

This function of x is of such a form as to be a maximum for

$$x = (p_m \div p_n) = (\tfrac{4}{5})^3 = .512; \quad . \quad . \quad . \quad . \quad (2)$$

i.e., theoretically, if the *state n* inside the reservoir remains the same, while the outside tension (considered $= p_m$ of jet, Fig. 619) is made to assume lower and lower values (so that $x, = p_m \div p_n$, diminishes in the same ratio), the maximum flow of weight per unit of time will occur when $p_m = .512\ p_n$, a little more than half the inside tension. (With the more accurate value 1.41 (1.408), instead of $\frac{3}{2}$, see § 478, we should obtain .527 instead of .512 for dry air; see § 549.)

Prof. Cotterill says (p. 544 of his "Applied Mechanics"): "The diminution of the theoretical discharge on diminution of the external pressure below the limit just now given is an anomaly which had always been considered as requiring explanation, and M. St. Venant had already suggested that it could not actually occur. In 1866 Mr. R. D. Napier showed by experiment that the weight of steam of given pressure discharged from an orifice really is independent of the pressure of the medium into which efflux takes place*; and in 1872 Mr. Wilson confirmed this result by experiments on the reaction of steam issuing from an orifice."

"The explanation lies in the fact that the pressure in the centre of the contracted jet is not the same as that of the surrounding medium. The jet after passing the contracted section suddenly expands, and the change of direction of the fluid particles gives rise to centrifugal forces" which cause the pressures to be greater in the centre of the contracted section than at the circumference; see Fig. 620.

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* When the difference between internal and external pressures is great,—should be added.

Prof. Cotterill then advises the assumption that $p_m = .527 p_n$ (for air and perfect gases) as the mean tension in the jet at m (Fig. 619), *whenever the outside medium is at a tension less than* $.527 p_n$. He also says, "Contraction and friction must be allowed for by the use of a coefficient of discharge the value of which, however, is more variable than that of the corresponding coefficient for an incompressible fluid. Little is certainly known on this point." See §§ 549 and 554.

For air the velocity of this *maximum flow of weight* is

$$Vel.\ of\ max.\ G = \left[997 \sqrt{\frac{T_n}{T_0}}\right] \text{ ft. per sec.,} \quad . \quad (3)$$

where T_n = abs. temp. in reservoir, and T_0 = that of freezing point. Rankine's Applied Mechanics (p. 584) mentions experiments of Drs. Joule and Thomson, in which the circular orifices were in a thin plate of copper and of diameters 0.029 in., 0.053 in., and 0.084 in., while the outside tension was about one half of that inside. The results were 84 per cent of those demanded by theory, a discrepancy due mainly, as Rankine says, to the fact that the actual area of the orifice was used in computation instead of the contracted section; i.e., contraction was neglected.

554. Coefficients of Efflux by Experiment. For Orifices and Short Pipes. Small Difference of Tensions.—Since the discharge through an orifice or short pipe from a reservoir is affected not only by contraction, but by slight friction at the edges, even with a rounded entrance, the theoretical results for the volume and weight of flow per unit of time in preceding paragraphs should be multiplied both by a coefficient of velocity ϕ and one for contraction C, as in the case of water; i.e., by a *coefficient of efflux* $\mu, = \phi C$. (Of course, when there is no contraction, $C = 1.00$, and then $\mu = \phi$ as with a well-rounded mouth-piece, for instance, Fig. 541, and with short pipes.)

Hence for practical results, with orifices and short pipes, we should write for the *weight of flow per unit of time*

$$= G = \mu F v_m \gamma_m = \mu F \left(\frac{p_m}{p_n}\right)^{\frac{2}{3}} \sqrt{2g3p_n\gamma_n\left[1 - \left(\frac{p_m}{p_n}\right)^{\frac{1}{3}}\right]} \quad . \quad (1)$$

(from the equations of § 552 for adiabatic flow, as most accurate; $p_m \div p_n$ may range from $\frac{1}{2}$ to 1.00). $F =$ area of orifice, or of discharging end of mouth-piece or short pipe. $\gamma_n =$ heaviness of air in reservoir and $= T_0 p_n \gamma_0 \div T_n p_0$, eq. (13) of § 437; and $\mu =$ the experimental coefficient of efflux.

From his own experiments and those of Koch, D'Aubuisson, and others, Weisbach recommends the following mean values of μ for various mouthpieces, when p_n is not more than $\frac{1}{6}$ larger than p_m (i.e., about 17 % larger), for use in eq. (1):

1. For an orifice in a thin plate, ... $\mu = 0.56$
2. For a short cylindrical pipe (inner corners not rounded), $\mu = 0.75$
3. For a well-rounded mouth-piece (like that in Fig. 541), $\mu = 0.98$
4. For a short conical convergent pipe (angle about 6°), $\mu = 0.92$

EXAMPLE.—(Data from Weisbach's Mechanics.) "If the sum of the areas of two conical tuyères of a blowing-machine is $F = 3$ sq. inches, the temperature in the reservoir 15° Cent., the height of the attached (open) mercury manometer (see Fig. 464) 3 inches, and the height of the barometer in the external air 29 inches," we have (ft., lb., sec.)

$$\frac{p_m}{p_n} = \frac{29}{29 + 3} = \frac{29}{32}; \quad T_n = 288° \text{ Abs. Cent.};$$

$$p_n = \left(\tfrac{32}{30}\right) 14.7 \times 144 \text{ lbs. per sq. ft.};$$

$$\gamma_n = \tfrac{273}{288} \cdot \tfrac{32}{30} \times 0.0807 = 0.0816 \text{ lbs. per cub. ft.},$$

while $F = \frac{3}{144}$ sq. ft. and (see above) $\mu = 0.92$; hence

$$G = 0.92 \times \tfrac{3}{144} \left(\tfrac{29}{32}\right)^{\frac{2}{3}} \sqrt{2 \times 32.2 \times 3 \times \tfrac{32}{30} \times 14.7 \times 144 \times .0816 \left[1 - \sqrt[3]{\tfrac{29}{32}}\right]};$$

i.e., $G = .6076$ lbs. per second; which will occupy a volume

$$V_0 = G \div \gamma_0 = G \div .0807 = 7.59 \text{ cub. ft.}$$

at one atmosphere tension and freezing-point temperature, while at a temperature of $T_n = 288°$ Abs. Cent. and tension of $p_m = \frac{29}{30}$ of one atmosphere (i.e., in the state in which it was on entering the blowing-engine) it occupied a volume

$$V = \tfrac{288}{273} \cdot \tfrac{30}{29} \times 7.59 = 8.24 \text{ cub. ft.}$$

(This last is Weisbach's result, obtained by an approximate formula.)

555. Coefficients of Efflux for Orifices and Short Pipes for a Large Difference of Tension.—For values $> \frac{7}{6}$ and < 2, of the ratio $p_n : p_m$, of internal to external tension, Weisbach's experiments with *circular orifices in thin plate*, of diameters ($= d$) from 0.4 inches to 0.8 inches, gave the following results:

$p_n : p_m =$	1.05	1.09	1.40	1.65	1.90	2.00
for $d = .4^{\text{in.}}$; $\mu =$	.55	.59	.69	.72	.76	.78
" $d = .8^{\text{in.}}$; $\mu =$	.56	.57	.64	.68		.72

Whence it appears that μ increases somewhat with the ratio of p_n to p_m, and decreases slightly for increasing size of orifice.

With short cylindrical pipes, internal edges not rounded, and three times as long as wide, Weisbach obtained μ as follows:

$p_n : p_m =$	1.05	1.10	1.30	1.40	1.70	1.74
diam. $= .4^{\text{in.}}$; $\mu =$	.73	.77	.83			
" $= .6^{\text{in.}}$; $\mu =$				.81	.82	
" $= 1.0^{\text{in.}}$; $\mu =$						.83

When the inner edges of the 0.4 in. pipe were slightly rounded, μ was found $= 0.93$; while a well-rounded mouthpiece of the form shown in Fig. 541 gave a value $\mu =$ from .965 to .968, for $p_n : p_m$ ranging from 1.25 to 2.00. These values of μ are for use in eq. (1), above.

556. To find the Discharge when the Internal Pressure is measured in a Small Reservoir or Pipe, not much larger than the

Orifice.—Fig. 621. If the internal pressure p_n, and temperature T_n, must be measured in a small reservoir or pipe, n, whose sectional area F_n is not very large compared with that of the orifice, F, (or of the jet, F_m,) the velocity v_n at n (velocity of approach) cannot be put = zero. Hence, in applying eq. (B), § 550, to the successive laminæ between n and m, and integrating, we shall have, for *adiabatic steady flow*,

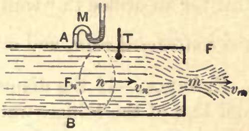

FIG. 621.

$$\frac{v_m^2}{2g} - \frac{v_n^2}{2g} = \frac{3p_n}{\gamma_n}\left[1 - \left(\frac{p_m}{p_n}\right)^{\frac{1}{3}}\right] \quad . \quad . \quad . \quad . \quad (1)$$

instead of eq. (1) of § 552. But from the equation of continuity for steady flow of gases [eq. (a) of § 548], $F_n v_n \gamma_n = F_m v_m \gamma_m$; hence $v_n^2 = \frac{F_m^2 \gamma_m^2}{F_n^2 \gamma_n^2} v_m^2$, while for an adiabatic change from n to m, $\frac{\gamma_m}{\gamma_n} = \left(\frac{p_m}{p_n}\right)^{\frac{2}{3}}$; whence by substitution in (1), solving for v_m, we have

$$v_m = \frac{\sqrt{2g \cdot \frac{3p_n}{\gamma_n}\left[1 - \left(\frac{p_m}{p_n}\right)^{\frac{1}{3}}\right]}}{\sqrt{1 - \left(\frac{F_m}{F_n}\right)^2 \left(\frac{p_m}{p_n}\right)^{\frac{4}{3}}}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As before, from §§ 472 and 478,

$$\gamma_n = \frac{p_n}{p_0} \frac{T_0}{T_n} \cdot \gamma_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\gamma_m = \left(\frac{p_m}{p_n}\right)^{\frac{2}{3}} \gamma_n \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Having observed p_n, p_m, and T_n, then, and knowing the area F of the orifice, we may compute γ_n, γ_m, and v_m, and finally the

$$\textit{Weight of flow per time-unit} = G = \mu F v_m \gamma_m, \quad . \quad . \quad (5)$$

taking μ from § 554 or 555. In eq. (2) it must be remembered that for an orifice in "thin plate," F_m is the sectional area of the *contracted vein*, and $= CF$; where C may be put $= \frac{\mu}{.97}$.

EXAMPLE.—If the diameter of AB, Fig. 621, is $3\frac{1}{2}$ inches, and that of the orifice, well rounded, $= 2$ in.; if $p_n = 1\frac{1}{12}$ atmospheres $= \frac{13}{12} \times 14.7 \times 144$ lbs. per sq. ft., while $p_m = \frac{11}{12}$ of an atmos., so that $\frac{p_m}{p_n} = \frac{11}{13}$, and $T_n = 283°$ Abs. Cent.,—required the discharge per second, using the ft., lb., and sec.

From eq. (3),

$$\gamma_n = \tfrac{13}{12} \cdot \tfrac{273}{283} \times 0.0807 = .08433 \text{ lbs. per cub. ft.};$$

whence (eq. (4))

$$\gamma_m = (\tfrac{11}{13})^{\frac{2}{3}} \gamma_n = .07544 \text{ lbs. per cub. ft.}$$

Then, from eq. (2),

$$v_m = \left[\sqrt{\frac{64.4 \times 3 \times 15.925 \times 144}{.08433}\left(1 - (\tfrac{11}{13})^{\frac{1}{3}}\right)}\right] \div \left[\sqrt{1 - (\tfrac{16}{49})^2 (\tfrac{11}{13})^{\frac{4}{3}}}\right]$$

$$= 558.1 \text{ ft. per sec.};$$

$$\therefore\ G = 0.98\,\frac{\pi}{4}\left(\frac{1}{6}\right)^2 558.1 \times .07544 = .9003 \text{ lbs. per sec.}$$

557. Transmission of Compressed Air; through very Long Level Pipes. Steady Flow.

CASE I. *When the difference between the tensions in the reservoirs at the ends of the pipe is small.*—Fig. 622. Under these circumstances it is simpler to employ the form of formula that would be obtained for a liquid by applying Bernoulli's Theorem, taking into account the "loss of head" occasioned

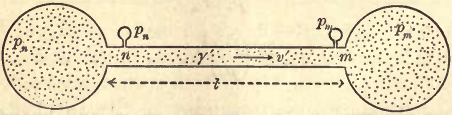

FIG. 622.

by the friction on the sides of the pipe. Since the pipe is very long, and the change of pressure small, the mean velocity in the pipe, v', assumed to be nearly the same at all points along the pipe, will not be large; hence the difference between the velocity-heads at n and m will be neglected; a certain *mean heaviness* γ' will be assigned to all the gas in the pipe, as if a liquid.

Applying Bernoulli's Theorem, *with friction*, § 516, to the ends of the pipe, n and m, we have (as for a liquid)

$$\frac{v_m^2}{2g} + \frac{p_m}{\gamma'} + 0 = \frac{v_n^2}{2g} + \frac{p_n}{\gamma'} + 0 - 4f\frac{l}{d}\frac{v'^2}{2g}. \quad . \quad (1)$$

Putting (as above mentioned) $v_m^2 - v_n^2 = 0$, we have, more simply,

$$\frac{p_n - p_m}{\gamma'} = 4f\frac{l}{d}\cdot\frac{v'^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of f as *coefficient of friction for air in long pipes* is found to be somewhat smaller than for water; see next paragraph.

558. Transmission of Compressed Air. Experiments in the St. Gothard Tunnel, 1878.—[See p. 96 of Vol. 24 (Feb. '81), Van Nostrand's Engineering Magazine.] In these experiments, the temperature and pressure of the flowing gas (air) were observed at each end of a long portion of the pipe which delivered the compressed air to the boring-machines three miles distant from the tunnel's mouth. The portion considered was selected at a distance from the entrance of the tunnel, to eliminate the fluctuating influence of the weather on the temperature of the flowing air. A *steady flow* being secured by proper regulation of the compressors and distributing tubes, observations were made of the internal pressure (p), internal temperature (T), as well as the external, at each end of the portion of pipe considered, and also at intermediate points; also of the *weight of flow per second* $G = Q_0\gamma_0$, measured at the compressors under standard conditions (0° Cent. and one atmos. tension). Then knowing the p and T at any section of the pipe, the heaviness γ of the air passing that section can be computed $\left[\text{from } \frac{\gamma}{\gamma_0} = \frac{p}{p_0} \cdot \frac{T_0}{T}\right]$ and the velocity $v = G \div F\gamma$, F being the sectional area at that point. Hence the *mean velocity* v', and the *mean heaviness* γ', can be computed for this portion of the pipe whose diameter $= d$ and length $= l$. In the experiments cited it was found that at points not too near the tunnel-mouth the temperature inside the pipe was always about 3° Cent. lower than that of the tunnel. The values of f in the different experiments were then computed from eq. (2) of the last paragraph; i.e.,

$$\frac{p_n - p_m}{\gamma'} = 4f\frac{l}{d} \cdot \frac{v'^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

all the other quantities having been either directly observed, or computed from observed quantities.

THE ST. GOTHARD EXPERIMENTS.

[*Concrete quantities reduced to English units.*]

No.	l (feet.)	d (ft.)	γ (lbs. cub. ft.)	Atmospheres.		$p_n - p_m$ lbs. sq. in.	v' ft. per sec.	mean temp. Cent.	f
				p_n	p_m				
1	15092	$\frac{2}{3}$	0.4058	5.60	5.24	5.29	19.32	21°	.0035
2	15092	$\frac{2}{3}$	0.3209	4.35	4.13	3.23	16.30	21°	.0038
3	15092	$\frac{2}{3}$	.2803	3.84	3.65	2.79	15.55	21°	.0041
4	1712	$\frac{1}{2}$	.3765	5.24	5.00	3.52	37.13	26.5	.0045
5	1712	$\frac{1}{2}$	.3009	4.13	4.06	1.03	30.82	26.5	.0024(?)
6	1712	$\frac{1}{2}$	.2641	3.65	3.54	1.54	29.34	26.5	.0045

In the article referred to (Van Nostrand's Mag.) f is not computed. The writer contents himself with showing that Weisbach's values (based on experiments with small pipes and high velocities) are much too great for the pipes in use in the tunnel.

With small tubes an inch or less in diameter Weisbach found, for a velocity of about 80 ft. per second, $f = .0060$; for still higher velocities f was smaller, approximately, in accordance with the relation

$$f = \frac{.0542}{\sqrt{v'} \text{ (in ft. per sec.)}}$$

On p. 370, vol. XXIV, Van Nostrand's Mag., Prof. Robinson of Ohio mentions other experiments with large long pipes.

From the St. Gothard experiments a value of $f = .004$ may be inferred for approximate results with pipes from 3 to 8 in. in diameter.

EXAMPLE.—It is required to transmit, in steady flow, a supply of $G = 6.456$ lbs. of atmospheric air per second through a pipe 30000 ft. in length (nearly six miles) from a reservoir where the tension is 6.0 atmos. to another where it is 5.8 atmos., the mean temperature in the pipe being 80° Fahr., = 24° Cent. (i.e. = 297° Abs. Cent.). Required the proper diameter of pipe; $d = ?$ The value $f = .00425$ will be used, and the ft.-lb.-sec. system of units. The mean volume passing per second in the pipe is

$$Q' = G \div \gamma'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The mean velocity may thus be written: $v' = \frac{Q'}{F} = \frac{Q'}{\frac{1}{4}\pi d^2}$. (4)

The mean heaviness of the flowing air, computed for a mean tension of 5.9 atmospheres, is, by § 472,

$$\gamma' = \frac{5.9 \times 14.7}{1 \times 14.7} \cdot \frac{273}{297} \times .0807 = 0.431 \text{ lbs. per cub. ft.};$$

and hence, see eq. (3),

$$Q' = \frac{G}{\gamma'} = \frac{6.456}{0.431} = 14.74 \text{ cub. ft.}$$

at tension of 5.9 atmos., and temperature 297° Abs. Cent.

Now, from eq. (2),

$$\frac{p_n - p_m}{\gamma'} = \frac{4f}{2g} \cdot \frac{l}{d} \cdot \left[\frac{Q'}{\frac{1}{4}\pi d^2}\right]^2;$$

whence

$$d^5 = \frac{4f}{(\frac{1}{4}\pi)^2} \cdot \frac{\gamma' l}{(p_n - p_m)} \cdot \frac{Q'^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and hence, numerically,

$$d = \sqrt[5]{\frac{4 \times .00425 \times 0.431 \times 30000 \times (14.74)^2}{(.7854)^2[14.7 \times 144(6.00 - 5.80)]2 \times 32.2}} = 1.23 \text{ feet.}$$

559. (Case II of § 557) Long Pipe, with Considerable Difference of Pressure at Extremities of the Pipe. Flow Steady.—Fig. 623. If the difference between the end-tensions is comparatively *great,* we can no longer deal with the whole of the air in the pipe at once, as regards ascribing to it a mean velocity and mean tension, but *must consider the separate laminæ,* such as AB (a short length of the air-stream) to which we may apply eq. (2) of § 557; A and B corresponding to the n and m of Fig. 622. Since the $p_n - p_m$, l, γ', and v' of § 557 correspond to the $-dp$, ds, γ, and v of the present case (short section or lamina), we may write

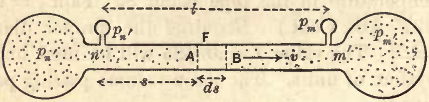

FIG. 623.

$$-\frac{dp}{\gamma} = 4f\frac{v^2}{d\,2g}\,ds. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But if G = weight of flow per unit of time, we have at any section, $Fv\gamma = G$ (equation of continuity); i.e., $v = G \div F\gamma$, whence by substitution in eq. (1) we have

$$-\frac{dp}{\gamma} = \frac{4f}{2gd}\cdot\frac{G^2ds}{F^2\gamma^2}; \quad \text{i.e.,} \quad -\gamma dp = \frac{4fG^2}{2gF^2d}\,ds. \quad . \quad (2)$$

Eq. (2) contains *three variables,* γ, p, and s (= distance of lamina from n'). As to the dependence of the heaviness γ on the tension p in different laminæ, experiment shows that in most cases a uniform temperature is found to exist all along the pipe, if properly buried, or shaded from the sun; the loss of heat by adiabatic expansion being in great part made up by the heat generated by the friction against the walls of the pipe. This is due to the small loss of tension per unit of length of pipe as compared with that occurring in a short discharge pipe or nozzle. Hence we may treat the flow as *isothermal*, and write $p \div \gamma = p_{n'} \div \gamma_{n'}$ (§ 475, Mariotte's Law). Hence $\gamma = \frac{\gamma_{n'}}{p_{n'}}p$, which substituted in eq. (2) enables us to write:

$$-pdp = \left[\frac{4fG^2}{2gF^2d}\frac{p_{n'}}{\gamma_{n'}}\right]ds. \quad . \quad . \quad . \quad . \quad . \quad (3)'$$

$$\therefore\ -\int_{n'}^{m'} pdp = \left[\frac{4fG^2p_{n'}}{2gF^2\gamma_{n'}d}\right]\int_0^l ds. \quad . \quad . \quad . \quad . \quad (3)$$

Performing the integration, noting that at n' $p = p_{n'}$, $s = 0$, and at m' $p = p_{m'}$ and $s = l$, we have

$$\tfrac{1}{2}[p_{n'}^2 - p_{m'}^2] = \frac{4fl}{2gd}\cdot\frac{G^2}{F^2}\cdot\frac{p_{n'}}{\gamma_{n'}}\cdot\cdot \left\{\begin{matrix}\textit{isothermal flow}\\ \textit{in long pipes}\end{matrix}\right\} \quad . \quad . \quad (4)$$

It is here assumed that the tension at the entrance of the pipe is practically equal to that in the head reservoir, and that at the end (m') to that of the receiving reservoir; which is not strictly true, especially when the corners are not rounded. It will be remembered also that in establishing eq. (2) of § 557 (the basis of the present paragraph), the "inertia" of the gas was neglected; i.e., the change of velocity in passing along the pipe. Hence eq. (4) should not be applied to cases where the pipe is so short, or the difference of end-tensions so great, as to create a considerable difference between the velocities at the two ends of the pipe. (See Addendum on p. 797.)

EXAMPLE.—A well or reservoir supplies natural gas at a tension of $p_{n'} = 30$ lbs. per sq. inch. Its heaviness at 0° Cent. and one atmosphere tension is .0484 lbs. per cub. foot. In piping this gas along a level to a town two miles distant, a single four-inch pipe is to be employed, and the tension in the receiving reservoir (by proper regulation of the gas distributed from it) is to be kept equal to 16 lbs. per sq. in. (which would sustain a column of water about 2 ft. in height in an *open* water manometer, Fig. 465).

The mean temperature in the pipe being 17° Cent., required the amount (weight) of gas delivered per second, supposing leakage to be prevented (formerly a difficult matter in practice). Solve (4) for G, and we have

$$G = \tfrac{1}{4}\pi d^2\sqrt{\frac{gd}{4fl}\cdot\frac{\gamma_{n'}}{p_{n'}}(p_{n'}^2 - p_{m'}^2)}. \quad . \quad . \quad . \quad (5)$$

First, from § 472, with $T_{n'} = T_{m'} = 290°$ Abs. Cent., we compute

$$\frac{p_{n'}}{\gamma_{n'}} = \frac{p_0}{\gamma_0}\cdot\frac{T_{n'}}{T_0} = \frac{14.7 \times 144}{.0484}\cdot\frac{290}{273} = 46454 \text{ feet.}$$

Hence with $f = .005$,

$$G = \tfrac{1}{4}\pi(\tfrac{4}{12})^2\cdot\sqrt{\frac{32.2 \times \frac{4}{12}[(30 \times 144)^2 - (16 \times 144)^2]}{4 \times .005 \times 10560 \times 46454}}$$

$$= 0.337 \text{ lbs. per sec.}$$

(For compressed atmospheric air, under like conditions, we would have $G = 0.430$ lbs. per second.)

Of course the proper choice of the coefficient f has an important influence on the result.

From the above result ($G = 0.337$ lbs. per second) we can compute the volume occupied by this quantity of gas in the receiving reservoir, using the relation $Q_{m'} = \dfrac{G}{\gamma_{m'}}$.

The heaviness $\gamma_{m'}$ of the gas in the receiving reservoir is most easily found from the relation $\dfrac{p_{m'}}{\gamma_{m'}} = \dfrac{p_{n'}}{\gamma_{n'}}$, which holds good since the flow is *isothermal*. I.e., $\dfrac{p_{m'}}{\gamma_{m'}} = 46454$ ft.; whence $\gamma_{m'} = 0.049$ lbs. per cubic foot, $p_{m'}$ being 16×144 lbs. per sq. ft.

Hence

$$Q_{m'} = \frac{G}{\gamma_{m'}} = \frac{0.337}{0.049} = 6.794 \text{ cub. ft. per sec.}$$

It should be said that the pressure at the up-stream end of the pipe depends upon the rate of flow allowed to take place.

With no flow permitted, the pressure in the tube of a gas-well has in some cases reached the high figure of 500 or 600 lbs. per sq. in.

560. Rate of Decrease of Pressure along a Long Pipe.—Considering further the case of the last paragraph, that of a straight, long, level pipe of uniform diameter, delivering gas from a storage reservoir into a receiving reservoir, we note that if in eq. (4) we retain $p_{m'}$ to indicate the tension in the receiving reservoir, but let $p_{n'}$ denote in turn the tension at points in the pipe successively further and further (a distance x) from the receiving reservoir m', we may write x for l and obtain the equation (between two variables, $p_{n'}$ and x)

$$p_{n'}^2 - p_{m'}^2 = Const. \times x. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This can be used to bring out an interesting relation mentioned by a writer in the *Engineering News* of July 1887 (p. 71), viz., the fact that in the parts of the pipe more distant from the receiving end, m', the distance along the pipe in which a given loss of pressure occurs is much greater than near the receiving end.

To make a numerical illustration, let us suppose that the pipe is of such size, in connection with other circumstances, that the tension $p_{n'}$ at A, a distance $x =$ six miles from m', is two atmospheres, the tension in the receiving reservoir being one atmosphere; that is, that the loss of tension between A and m' is one atmosphere. If we express tensions in atmospheres and distances in miles, we have for the value of the constant in eq. (6), for this case,

$$\text{Const.} = (4 - 1) \div 6 = \tfrac{3}{6}\,; \quad (\textit{for assumed units.}) \quad . \quad . \quad (7)$$

Now let $p_{n'} =$ the tension at B, a point 18 miles from m', and we have, from eqs. (6) and (7), the tension at $B = 3.16$ atmospheres. Proceeding in this manner, the following set of values is obtained:

Point.	Total distance from m'.	Distance between consecutive points.	Tension at point.	Loss of tension in each interval.
F	126 miles.	36 miles.	8.00 atm.	1.22 atm.
E	90 "	30 "	6.78 "	1.22 "
D	60 "	24 "	5.56 "	1.21 "
C	36 "	18 "	4.35 "	1.19 "
B	18 "	12 "	3.16 "	1.16 "
A	6 "	6 "	2.00 "	1.00 "
m'	0 "	....	1.00 "	....

If the distances and tensions in the second and fourth columns be plotted as abscissæ and ordinates of a curve, the latter is a parabola with its axis following the axis of the pipe: its vertex is not at m', however.

561. Long Pipe of Variable Diameter.—Another way of stating the fact mentioned in the last paragraph is as follows: At the up-stream end of the pipe of *uniform diameter* the gas is of much greater density than at the other extremity (the heaviness is directly as the tension, the temperature being assumed the same throughout the pipe), and the velocity of its motion is smaller than at the discharging end (in the same ratio). It is true that the frictional resistance per unit of length of pipe varies directly as the heaviness [eq. (1), § 510], but also true that it varies as the *square* of the velocity; so that, for instance, if the pressure at a point A is double that at B in the pipe of constant diameter, it implies that the heaviness and velocity at A are double and half, respectively, those at B, and thus the gas at A is subjected to only half the frictional resisting force per foot of length as compared with that at B. Hence the relatively small diminution, per unit of length, in the tension at the up-stream end in the example of the last paragraph.

In the pipe of uniform diameter, as we have seen, the greater part of the length is subjected to a comparatively high tension, and is thus under a greater liability to loss by leakage than if the decrease of tension were more uniform. The total "*hoop-tension*" (§ 426) in a unit length of pipe, also, is proportional to the gas tension, and thinner walls might be employed for the down-stream portions of the pipe if the gas tension in those portions could be made smaller than as shown in the preceding example.

To secure a more rapid fall of pressure at the up-stream end of the pipe, and at the same time provide for the same delivery of gas as with a pipe of uniform diameter throughout, a pipe of *variable* diameter may be employed, that diameter being considerably smaller at the inlet than that of the uniform pipe but progressively enlarging down-stream. This will require the diameters of portions near the discharging end to be larger than in the uniform pipe, and if the same thickness of metal were necessary throughout, there would be no saving of metal, but rather the reverse, as will be seen; but the diminished thickness made practicable in those parts from a less total hoop tension than in the corresponding parts of the uniform pipe more than compensates for the extra metal due to increased circumference, aside from the diminished liability to leakage, which is of equal importance.

A simple numerical example will illustrate the foregoing. The pipe being circular, we may replace F by $\frac{1}{4}\pi d^2$ in equation (4), and finally derive, G being given,

$$d = Const. \times \left[\frac{l}{p_{n'}^2 - p_{m'}^2}\right]^{\frac{1}{5}} = C. \left[\frac{l}{p_{n'}^2 - p_{m'}^2}\right]^{\frac{1}{5}}. \quad . \quad (8)$$

Let A be the head reservoir, and m' the receiving reservoir, and B a point half-way between. At A the tension is 10 atmospheres; at m', 2 atmospheres. For transmitting a given weight of gas per unit-time, through a pipe of constant diamter throughout, that diameter must be (tensions in atmospheres; $2l_0$ being the length), by eq. (8),

$$d = Cl_0^{\frac{1}{5}} \left[\frac{2}{100 - 4}\right]^{\frac{1}{5}} = Cl_0^{\frac{1}{5}}(.0208)^{\frac{1}{5}} = 0.46\, Cl_0^{\frac{1}{5}}. \quad . \quad (8)_1$$

If we substitute for the pipe mentioned, another having a constant diameter d_1 from A to B, where we wish the tension to be 5 atmospheres, and a different constant diameter d_2 from B to m', we derive similarly

$$d_1 = Cl_0^{\frac{1}{5}}\left[\frac{1}{100 - 25}\right]^{\frac{1}{5}} = 0.42\, Cl_0^{\frac{1}{5}}$$

and

$$d_2 = Cl^{\frac{1}{5}}\left[\frac{1}{25-4}\right]^{\frac{1}{5}} = 0.54\ Cl^{\frac{1}{5}}$$

It is now to be noted that the sum of d_1 and d_2 is slightly greater than the double of d; so that if the same thickness of metal were used in both designs the compound pipe would require a little more material than the uniform pipe; but, from the reasoning given at the beginning of this paragraph, that thickness may be made considerably less in the down-stream part of the compound pipe, and thus economy secured.

[In case of a cessation of the flow, the gas tension in the whole pipe might rise to an equality with that of the head-reservoir were it not for the insertion, at intervals, of automatic regulators, each of which prevents the decrease of tension on its down-stream side below a fixed value. To provide for changes of length due to rise and fall of temperature, the pipe is laid with slight undulations.]

It is a noteworthy theoretical deduction that a given pipe of variable diameter connecting two reservoirs of gas at specified pressures will deliver the same weight of gas as before, *if turned end for end.* This follows from equation (3)′, § 559. With d variable, (3)′ becomes (with $F = \frac{1}{4}\pi d^2$)

$$\int_{n'}^{m'}(-p\,dp) = G^2C''\int_{n'}^{m'}\frac{ds}{d^5};\quad \text{i.e.,}\quad G^2 = \frac{p_{n'}^{\,2} - p_{m'}^{\,2}}{2C''\int_{n'}^{m'}\frac{ds}{d^5}}. \quad . \quad (9)$$

(C'' is a constant.)

But $\int_{n'}^{m'}\frac{ds}{d^5}$ is evidently the same in value if the pipe be turned end for end. In commenting on this circumstance, we should remember (see § 559) that the loss of pressure along the pipe is ascribed *entirely to frictional resistance*, and in no degree to changes of velocity (inertia).

On p. 73 of the *Engineering News* of July 1887 are given the following dimensions of a compound pipe in actual use, and delivering natural gas. The pressure in the head-reservoir is 319 lbs. per sq. in.; that in the receiving reservoir, 65. For 2.84 miles from the head-reservoir the diameter of the pipe is

8 in.; throughout the next 2.75 miles, 10 in.; while in the remaining 3.84 miles the diameter is 12 in. At the two points of junction the pressures are stated to be 185 and 132 lbs. per sq. in., respectively, during the flow of gas under the conditions mentioned.

561a. Values of the Coefficient of Fluid Friction for Natural Gas.—In the Ohio Report on Economic Geology for 1888 may be found an article by Prof. S. W. Robinson of the University of that State describing a series of interesting experiments made by him on the flow of natural gas from orifices and through pipes. By the insertion of Pitot tubes approximate measurements were made of the velocity of the stream of gas in a pipe. The following are some of the results of these experiments, $p_1 - p_2$ representing the loss of pressure (in lbs. per sq. inch) per mile of pipe-length, and f the coefficient of fluid friction, in experiments with a six-inch pipe:

$p_1 - p_2$	1.00	1.50	2.25	2.50	5.75	6.25
f	0.0025	0.0037	0.0052	0.0059	0.0070	0.0060

In the flow under observation Prof. Robinson concluded that f could be taken as approximately proportional to the fourth root of the cube of the velocity of flow; though calling attention to the fact that very reliable results could hardly be expected under the circumstances.

561b. [Addendum to § 559.] Isothermal Flow in a level pipe, with consideration of Inertia.—In eq. (1) of p. 697 neglect dz, put $w \div F = 4 \div d$, and divide through by v^2. In the second term put $G^2 \div F^2\gamma^2$ for v^2 and then $p(\gamma_m \div p_m)$ for γ. We now find the variables separated, and on integration for steady flow obtain (after putting $v_n \div v_m = F\gamma_m \div F\gamma_n = p_m \div p_n$),

$$\log_e\left(\frac{p_n}{p_m}\right) - \frac{g(p_n^2 - p_m^2)}{2} \cdot \frac{F^2}{G^2} \cdot \frac{\gamma_m}{p_m} = -\frac{2fl}{d}.$$

[Notation as in § 557 with $G = Fv\gamma$.]

CHAPTER IX.

IMPULSE AND RESISTANCE OF FLUIDS.

562. The so-called "Reaction" of a Jet of Water flowing from a Vessel.—In Fig. 624, if a frictionless but water-tight plug B be inserted in an orifice in the vertical side of a vessel mounted on wheels, the resultant action of the water on the rigid vessel (as a whole) consists of its weight G, and a force $P' = Fh\gamma$ (in which $F =$ the area of orifice) which is the excess of the horizontal hydrostatic pressures on the vessel wall toward the right (|| to paper) over those toward the left, since the pressure P, $= Fh\gamma$, exerted on the plug is felt by the post C, and not by the vessel. Hence the post D receives a pressure

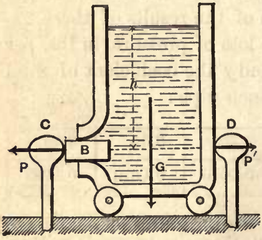

FIG. 624.

$$P' = Fh\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let the plug B be removed. A steady flow is then set up through the orifice, and now the pressure against the post D is $2Fh\gamma$ (as will be proved in the next paragraph); for not only is the pressure $Fh\gamma$ lacking on the left, because of the orifice, but the sum of all the horizontal components (|| to paper) of the pressures of the liquid filaments against the vessel wall around the orifice is less than its value before the flow began, by an amount $Fh\gamma$. A resistance $R = 2Fh\gamma$ being provided, and the post removed, a slow uniform motion may be maintained toward the right, the working force being $2Fh\gamma = P''$

(see Fig. 625; R is not shown). If an insufficient resistance be furnished before removing the post D, the vessel will begin to move toward the right with an *acceleration*, which will disturb the surface of the water and change the value of the horizontal force. This force

$$P'' = 2Fh\gamma \quad . \quad . \quad . \quad (2)$$

is called the "*reaction*" of the water-jet; γ is the heaviness of the liquid (§ 7).

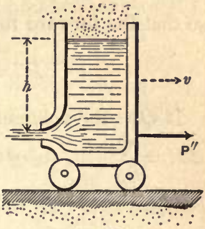

FIG. 625.

Of course, as the flow goes on, the water level sinks and the "reaction" diminishes accordingly. Looked upon as a motor, the vessel may be considered to be a piston-less and valve-less water-pressure engine, carrying its own reservoir with it.

In Case II of § 500 we have already had a treatment of the "Reaction-wheel" or "Barker's mill," which is a practical machine operating on this principle, and will be again considered in "Notes on Hydraulic Motors."

563. "Reaction" of a Liquid Jet on the Vessel from which it Issues.—Instead of showing that the pressures on the vessel close to the orifice are less than they were when there was no flow by an amount $Fh\gamma$ (a rather lengthy demonstration), another method will be given, of greater simplicity but somewhat fanciful.

If a man standing on the rear platform of a car is to take up in succession, from a basket on the car, a number of balls of equal mass $= M$, and project each one in turn horizontally backward with an acceleration $= p$, he can accomplish this only by exerting against each ball a pressure $= Mp$, and in the opposite direction against the car an equal pressure $= Mp$. If this action is kept up continuously the car is subjected to a constant and continuous forward force of $P'' = Mp$.

Similarly, the backward projection of the jet of water in the case of the vessel at rest must occasion a forward force against the vessel of a value dependent on the fact that in each small interval of time Δt a small mass ΔM of liquid has its velocity changed from zero to a backward velocity of $v = \sqrt{2gh}$; that is, has been projected with a mean acceleration of $p = \dfrac{v - 0}{\Delta t}$, so that the forward force against the vessel is

$$P'' = \text{mass} \times \text{acc.} = \frac{\Delta M . v}{\Delta t}. \quad . \quad . \quad . \quad . \quad (3)$$

If $Q =$ the volume of water discharged per unit time, then $\Delta M = \dfrac{Q\gamma}{g}\,\Delta t$, and since also $Q = Fv = F\sqrt{2gh}$, eq. (3) becomes

$$\text{“\textit{Reaction}” \textit{of jet}} = P'' = 2Fh\gamma. \quad . \quad . \quad . \quad (4)$$

(A similar proof, resulting in the same value for P'', is easily made if the vessel has a uniform motion with water surface horizontal.)

If the orifice is in "thin plate," we understand by F the area of the *contracted section*. Practically, we have $v = \phi\sqrt{2gh}$ (§ 495), and hence (3) reduces to

$$P'' = 2\phi^2 Fh\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Weisbach mentions the experiments of Mr. Peter Ewart of Manchester, England, as giving the result $P'' = 1.73Fh\gamma$ with a well-rounded orifice as in Fig. 625. He also found $\phi = .94$ for the same orifice, so that by eq. (4) we should have

$$P'' = 2(.94)^2 Fh\gamma = 1.77Fh\gamma.$$

With an orifice in thin plate Mr. Ewart found $P'' = 1.14Fh\gamma$. As for a result from eq. (4), we must put, for F, the area of the contracted section $.64F$ (§ 495), which, with $\phi = .96$, gives

$$P'' = 2(.96)^2 . 64Fh\gamma = 1.18Fh\gamma. \quad . \quad . \quad . \quad (6)$$

Evidently both results agree well with experiment.

Experiments made by Prof. J. B. Webb at the Stevens Institute (see Journal of the Franklin Inst., Jan. '88, p. 35) also confirm the foregoing results. In these experiments the vessel was suspended on springs and the jet directly downward, so that the "reaction" consisted of a diminution of the tension of the springs during the flow.

564. Impulse of a Jet of Water on a Fixed Curved Vane (with Borders).—The jet passes tangentially upon the vane. Fig.

626. B is the stationary nozzle from which a jet of water of cross-section F (area) and velocity $= c$ impinges tangentially upon the vane, which has plane borders, parallel to paper, to prevent the lateral escape of the jet. The curve of the vane is not circular necessarily. The vane being smooth, the velocity of the water in its curved path remains $= c$ at all points along the curve. Conceive the curve divided into a great number of small lengths each $= ds$, and subtending some angle $= d\phi$ from its own centre of curvature, its radius of curvature being $= r$ (different for different ds's), which makes some angle $= \phi$ with the axis Y (⌉ to original straight jet BA). At any instant of time there is an arc of water AD in contact with the vane, exerting pressure upon it. The pressure dP of any ds of the vane against the small mass of water $Fds\,.\,\gamma \div g$ then in contact with it is the "deviating" or "centripetal" force accountable for its motion in a curve of radius $= r$, and hence must have a value

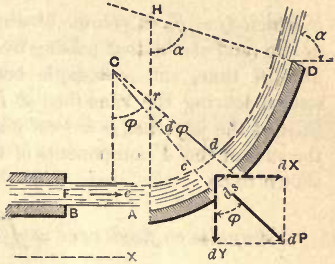

FIG. 626.

$$dP = \frac{F\gamma ds}{g} \cdot \frac{c^2}{r}. \quad . \quad . \quad (\S\ 76) \quad . \quad . \quad . \quad (1)$$

The opposite and equal of this force is the dP shown in Fig. 614, and is the impulse or pressure of this small mass against the vane. Its X-component is $dX = dP \sin \phi$. By making ϕ vary from 0 to α, and adding up the corresponding values of dX, we obtain the sum of the X-components of the small pressures exerted simultaneously against the vane by the arc of water then in contact with it; i.e., noting that $ds = rd\phi$,

$$\therefore \int_{\phi=0}^{\phi=\alpha} dX = \int_0^\alpha dP\,.\,\sin\phi = \frac{F\gamma c^2}{g}\int_0^\alpha \frac{ds\,.\,\sin\phi}{r}$$

$$= \frac{F\gamma c^2}{g}\int_0^\alpha [\sin\phi]d\phi = \frac{F\gamma c^2}{g}\Big[_0^\alpha - \cos\phi;$$

$$\left.\begin{array}{l}\textit{hence the X-impulse}\\ \textit{against fixed vane}\end{array}\right\} = \frac{F\gamma c^2}{g}[1-\cos\alpha] = \frac{Q\gamma c}{g}[1-\cos\alpha], \quad (2)$$

in which $Q = Fc =$ volume of water which passes through the nozzle (and also = that passing over the vane, in this case) per unit of time, and $\alpha =$ angle between the direction of the stream leaving the vane (i.e., at D) and its original direction (BA of the jet); i.e., $\alpha =$ *total angle of deviation.* Similarly, the sum of the Y-components of the dP's of Fig. 626 may be shown to be

$$\textit{Y-impulse on fixed vane} = \int_0^\alpha dP \cdot \cos\phi = \frac{Q\gamma c}{g}\sin\alpha \ldots (2)'$$

Hence the *resultant impulse* on the vane is a force

$$P'' = \sqrt{X^2 + Y^2} = \frac{Q\gamma c}{g}\sqrt{2(1-\cos\alpha)}, \quad . \quad . \quad (3)$$

and makes such an angle α', Fig. 627, with the direction BA, that

$$\tan\alpha' = \frac{Y}{X} = \frac{\sin\alpha}{1-\cos\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

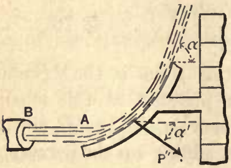

FIG. 627.

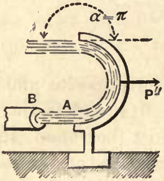

FIG. 628.

For example, if $\alpha = 90°$, then $\alpha' = 45°$; while if $\alpha = 180°$, Fig. 628, we have $\alpha' = 0°$; i.e., P'' is parallel to the jet BA, and its value is

$$P'' = 2Q\gamma\frac{c}{g}.$$

565. Impulse of a Jet on a Fixed Solid of Revolution whose Axis is Parallel to the Jet.—If the curved vane, with borders, of the preceding paragraph be replaced by a solid of revolution, Fig. 629, with its axis in line of the jet, the resultant pressure of the jet upon it will simply be the sum of the X-components (i.e., $= \text{to } BA$) of the pressures on all elements of the surface at a given instant; i.e.,

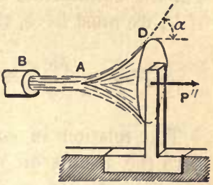

FIG. 629.

$$X = P'' = Q\gamma \frac{c}{g}(1 - \cos\alpha); \quad . \quad . \quad . \quad . \quad (5)$$

while the components ⌉ to X, all directed toward the axis of the solid, neutralize each other. For a *fixed plate*, then, Fig. 630, at right angles to the jet, we have for the force, or "impulse" (with $\alpha = 90°$),

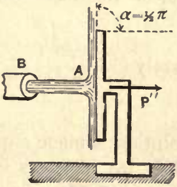

FIG. 630.

$$P'' = \frac{Q\gamma}{g}c = \frac{Fc^2}{g}\gamma = 2F\frac{c^2}{2g}\gamma. \quad . \quad (6)$$

The experiments of Bidone, made in 1838, confirm the truth of eq. (6) quite closely, as do also those of two students of the University of Pennsylvania at Philadelphia (see Jour. of the Frank. Inst. for Oct. '87, p. 258).

Eq. (6) is applicable to the theory of Pitot's Tube (see § 539), Fig. 631, if we consider the edge of the tube plane and quite wide. The water in the tube is at rest, and its section at A (of area $= F$) may be treated as a flat vertical plate receiving not only the hydrostatic pressure $Fx\gamma$, due to the depth x below the surface, but a continuous impulse $P'' = Fc^2\gamma \div g$ [see eq. (6)].

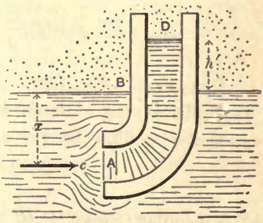

FIG. 631.

For the equilibrium of the end A, of the stationary column AD, we must have, therefore,

$$Fx\gamma + \frac{Fc^2\gamma}{g} = Fx\gamma + Fh'\gamma; \quad \text{i.e.,} \quad h' = (2.0)\frac{c^2}{2g}. \ . \ (7)$$

The relation in equation (7) corresponds reasonably well with the results of Weisbach's experiments with the instrument mentioned in § 539. Pitot himself, on trial of an instrument in which the edges of the tube at A were made flaring or conically divergent, like a funnel, found

$$h' = (1.5)\frac{c^2}{2g}; \ . \ . \ . \ . \ . \ . \ . \ . \ (7)'$$

while Darcy, desirous that the end of the tube should occasion as little disturbance as possible in the surrounding stream, made the extremity small and conically convergent.* The latter obtained the relation

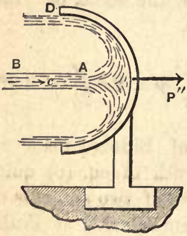

FIG. 632.

$$h' = \text{almost exactly } (1.0)\frac{c^2}{2g}. \ . \ (7)''$$

(See § 539.)

If the solid of revolution is made cup-shaped, as in Fig. 632, we have (as in Fig. 628) $\alpha = 180°$, and therefore, from eq. (5),

$$P'' = 2Q\gamma\frac{c}{g} = \frac{2Fc^2\gamma}{g} = 4F\left(\frac{c^2}{2g}\right)\gamma. \ . \ . \ . \ (8)$$

EXAMPLE.—Fig. 632. If $c = 30$ ft. per sec. and the jet (cylindrical) has a diameter of 1 inch, the liquid being water, so that $\gamma = 62.5$ lbs. per cub. ft., we have [ft., lb., sec.]

the impulse (force) $= P'' = \dfrac{2\frac{\pi}{4}\left(\frac{1}{12}\right)^2 900 \times 62.5}{32.2} = 19.05$ lbs.

Experiment would probably show a smaller result.

---

* See p. 833 for Mr. Freeman's Experiments.

566. Impulse of a Liquid Jet upon a Moving Vane having Lateral Borders and Moving in the Direction of the Jet.—Fig. 633. The vane has a motion of translation (§ 108) in the same direction as the jet. Call this the axis X. It is moving with a velocity v *away from the jet* (or, if toward the jet, v is negative). We consider v constant, its acceleration being prevented by a proper resistance (such as a weight $= G$) to balance the X-components of the arc-pressures. Before coming in contact with the vane, which it does tangentially (to avoid *sudden* deviation), the absolute velocity (§ 83) of the water in the jet $= c$, while its velocity relatively to the vane at A is $= c - v$; and it will now be proved that the relative velocity along the vane is constant. See Fig. 634. Let $v =$ the velocity of the vane (of each point of it, since its motion is one of translation), and $u =$ the velocity of a water particle (or small mass of water of length $= ds$) relatively to the point of the vane which it is passing. Then w, the absolute velocity of the small mass, is the diagonal formed on u and v. Neglecting friction, the only actual force acting on the mass is P, the pressure of the vane against it, and this is normal to the curve. Now an imaginary system of forces, equivalent to this actual system of one force P, i.e., capable of producing the same motion in the mass, may be conceived of, consisting of the individual forces which would produce, separately, the separate motions of which the actual motion of this small mass M is compounded. These component motions are as follows:

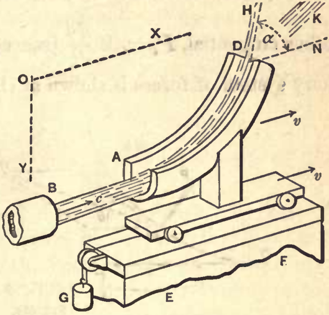

Fig. 633.

1. A horizontal uniform motion of constant velocity $= v$; and

2. A motion in the arc of a circle of radius $= r$ and with a

velocity $= u$, which we shall consider variable until proved otherwise.

Motion 1 is of such a nature as to call for *no force* (by Newton's first law of motion), while motion 2 could be maintained by a system of two forces, one normal, $P_n, = \frac{Mu^2}{r}$, and the other tangential, $P_t = M\frac{du}{dt}$ [see eq. (5), p. 76]. This imaginary system of forces is shown at (II.), Fig. 634, and is equivalent to the actual system at (I.). Therefore Σ(tang. compons.) in (I.) should be equal to Σ(tang. compons.) in (II.); whence we have

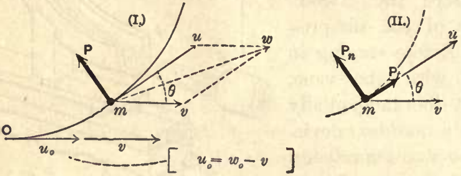

FIG. 634.

$$P_t = 0; \quad \text{i.e.,} \quad M\frac{du}{dt} = 0; \quad \text{or} \quad \frac{du}{dt} = 0; \quad . \quad . \quad (1)$$

i.e., u is constant along the vane and is equal to $c - v$ at every point. (The weight of the mass has been neglected since the height of the vane is small.) In Fig. 634 the symbol w_0 has been used instead of c, and the point 0 corresponds to A in Fig. 633.

[N.B. If the motion of the vane were *rotary*, about an axis ⌉ to AB (or to c), this relative velocity would be different at different points. See Notes on Hydraulic Motors. If the radius of motion of the point A, however, is quite large compared with the projection of AD upon this radius. the relative velocity is approximately $= c - v$ at all parts of the vane, and will be taken $= c - v$ in treating the "Hurdy gurdy" in § 567.]

By putting Σ(normal compons.) of (I.) = Σ(normal compons.) in (II.) we have

$$P = P_n; \quad \text{i.e.,} \quad P = M\frac{u^2}{r} = \frac{M(c-v)^2}{r}; \quad . \quad . \quad . \quad (2)$$

so that to find the sum of the X-components of the pressures exerted against the vane simultaneously by all the small masses of water in contact with it at any instant, the analysis differs from that in § 564 only in replacing the c of that article by the $(c - v)$ of this. Therefore

$$\Sigma(X\text{-}pressures) = P_x = \frac{F\gamma}{g}(c-v)^2[1 - \cos\alpha], \quad . \quad (3)$$

(where α is the angle of total deviation, relatively to vane, of the stream leaving the vane, from its original direction), and is seen to be proportional to the *square* of the relative velocity. F is the sectional area of jet, and γ the heaviness (§ 7) of the liquid. The Y-component (or P_Y) of the resultant impulse is counteracted by the support EF, Fig. 633. Hence, *for a uniform motion to be maintained*, with a given velocity $= v$, the weight G must be made $= P_x$ of eq. (3). (We here neglect friction and suppose the jet to preserve a practically horizontal direction for an indefinite distance before meeting the vane. If this uniform motion is to be *toward* the jet, v will be negative in eq. (3), making P_x (and $\therefore$ G) larger than for a positive v of same numerical value.

As to the *doing of work* [§§ 128, etc.], or exchange of energy, between the two bodies, jet and vane, during a uniform motion *away* from the jet, P_x exerts a *power* of

$$L = P_x v = \frac{F\gamma}{g}(c-v)^2 v[1 - \cos\alpha], \quad . \quad . \quad . \quad (4)$$

in which L denotes the number of units of work done per unit of time by P_x; i.e., the *power* (§ 130) exerted by P_x.

If v is negative, call it $-v'$, and we have the

$$\left.\begin{array}{l}\textit{Power expended}\\ \textit{by vane upon jet}\end{array}\right\} = P_x v' = \frac{F\gamma}{g}(c+v')^2 v'[1 - \cos\alpha]. \quad . \quad (5)$$

Of course, practically, we are more concerned with eq. (4) than with (5). The power L in (4) is a maximum for $v = \frac{1}{3}c$; but in practice, since a single moving vane or float cannot utilize the water of the jet as fast as it flows from the nozzle, let us conceive of a succession of vanes coming into position consecutively in front of the jet, all having the same velocity v; then the portion of jet intercepted between two vanes is at liberty to finish its work on the front vane, while additional work is being done on the hinder one; i.e., the water will be utilized as fast as it issues from the nozzle.

With such a *series of vanes*, then, we may put $Q', = Fc$, the volume of flow per unit of time from the nozzle, in place of $F(c - v) =$ the volume of flow per unit of time over the vane, in eq. (4); whence

$$\left.\begin{array}{r}\textit{Power exerted on}\\ \textit{series of vanes}\end{array}\right\} = L' = \frac{Q'\gamma}{g}[1 - \cos\alpha](c - v)v. \quad . \quad (6)$$

Making v variable, and putting $dL' \div dv = 0$, whence $c - 2v = 0$, we find that for $v = \frac{1}{2}c$, L', the power, is a maximum. Assuming different values for α, we find that for $\alpha = 180°$, i.e., by the use of a semicircular vane, or of a hemispherical cup, Fig. 635, with a point in middle, $1 - \cos\alpha$ is a max., $= 2$; whence, with $v = \frac{1}{2}c$, we have, as the *maximum power*,

$$L'_{\text{max.}} = \frac{Q'\gamma}{g}\cdot\frac{c^2}{2} = \frac{M'c^2}{2}; \left\{\begin{array}{l}\alpha = 180°,\\ v = \frac{1}{2}c;\end{array}\right\} ..(7)$$

in which M' denotes the mass of the flow per unit of time from the stationary nozzle. Now $\frac{M'c^2}{2}$ is the *entire kinetic energy* furnished per unit of time by the jet; hence the motor of Fig. 635 (*series of cups*) has a theoretical efficiency of unity, utilizing all the kinetic energy of the water. If this is true, the absolute velocity of the particles of liquid where they leave the cup, or vane, should be *zero*, which is seen to be true,

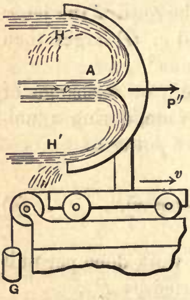

FIG. 635.

as follows: At H, or H', the velocity of the particles relatively to the vane is $= c - v =$ what it was at A, and hence is $= c - \frac{c}{2} = \frac{c}{2}$; hence at H the absolute velocity is $w =$ (rel. veloc. $\frac{c}{2}$ toward left)—(veloc. $\frac{c}{2}$ of vane toward right) $= 0$; Q.E.D. For $v >$ or $< \frac{1}{2}c$ this efficiency will not be attained.

567. The California "Hurdy-gurdy;" or Pelton Wheel.—The efficiency of unity in the series of cups just mentioned is in practice reduced to 80 or 85 per cent from friction and lateral escape of water. The Pelton wheel or California "Hurdy-gurdy," shown (in principle only) in Fig. 636, is designed to utilize the mechanical relation just presented, and yields results confirming the above theory, viz., that with the linear velocity of the cup-centres regulated to equal $\frac{c}{2}$, and with $\alpha = 180°$, the efficiency approaches unity or 100 per cent. Each cup has a projecting sharp edge or rib along the middle, to split the jet; see Fig. 635.

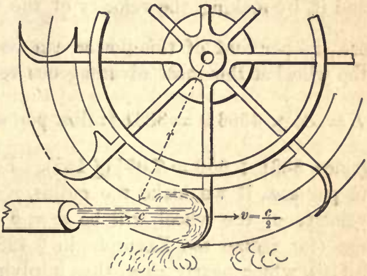

FIG. 636.

This wheel was invented to utilize small jets of very great velocities (c) in regions just deserted by "hydraulic mining" operators. Although c is great, still, by giving a large value to r, the radius of the wheel, the making of $v = \frac{c}{2}$ does not necessitate an inconveniently great speed of rotation (i.e., revolutions per unit of time). The plane of the wheel may be in any convenient position.

In the London *Engineer* of May '84, p. 397, is given an account of a test* made of a "Hurdy-gurdy," in which the motor showed an efficiency of 87 per cent. The diameter of the wheel was only 6 ft., that of the jet 1.89 in., and the head of the supply reservoir 386 ft., the water being transmitted through a pipe of 22 inches diameter and 6900 ft. in length. 107 H. P. was developed by the wheel.

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* See p. 834 for further details of this test and a perspective view of wheel.

EXAMPLE.—If the jet in Fig. 636 has a velocity $c = 60$ ft. per second, and is delivered through a 2-inch nozzle, the total power due to the kinetic energy of the water is (ft., lb., sec.)

$$\frac{Q'\gamma}{g} \cdot \frac{c^2}{2} = \frac{1}{32.2} \cdot \frac{\pi}{4}\left(\frac{2}{12}\right)^2 \times 60 \times 62.5 \times \tfrac{1}{2} \times 3600 = 4566.9 \left\{ \begin{array}{l} \text{ft. lbs.} \\ \text{p. sec.,} \end{array} \right.$$

and if, by making the velocity of the cups $= \frac{c}{2} = 30$ ft. per sec., 85 per cent of this power can be utilized, the power of the wheel at this most advantageous velocity is

$$L = .85 \times 4566.9 = 3881 \text{ ft. lbs. per sec.} = 7.05 \text{ horse-power}$$

[since $3881 \div 550 = 7.05$] (§ 132). For a cup-velocity of 30 ft. per sec., if we make the radius, r, = 10 feet, the angular velocity of the wheel will be $\omega = v \div r = 3.0$ *radians* per sec. (for radian see Example in § 428; for angular velocity, § 110), which nearly $= \pi$, thus implying nearly a half-revolution per sec.

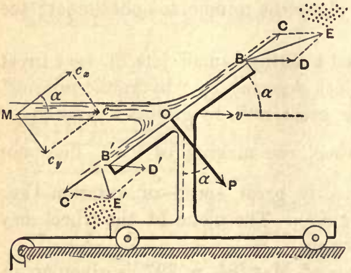


FIG. 637.

568. Oblique Impact of a Jet on a Moving Plate having no Border.—The plate has a motion of translation with a uniform veloc. $= v$ in a direction parallel to jet, whose velocity is $= c$. At O the filaments of liquid are deviated, so that in leaving the plate their particles are all found in the moving plane BB' of the plate surface, but the respective absolute velocities of these particles depend on the location of the point of the plate where they leave it, being found by forming a diagonal on the relative veloc. $c - v$ and the velocity v of the plate. For example, at B the absolute velocity of a liquid particle is

$$w = BE = \sqrt{v^2 + (c - v)^2 + 2v(c - v) \cos \alpha},$$

while at B' it is

$$w' = B'E' = \sqrt{v^2 + (c - v)^2 - 2v(c - v) \cos \alpha};$$

but evidently the component ⅂ to plate (the other component being parallel) of the absolute velocities of *all particles leaving* the plate, is the same and $= v \sin \alpha$. The skin-friction of the liquid on the plate being neglected, the resultant impulse of the jet against the plate must be *normal* to its surface, and its amount, P, is most readily found as follows:

Denoting by ΔM the mass of the liquid passing over the plate in a short time Δt, resolve the absolute velocities of all the liquid particles, before and after deviation, into components ⅂ to the plate (call this direction Y) and ‖ to the plate. Before meeting the plate the particles composing ΔM have a velocity in the direction of Y of $c_y = c \sin \alpha$; on leaving the plate a velocity in direction of Y of $v \sin \alpha$: they have therefore lost an amount of velocity in direction of $Y = (c - v) \sin \alpha$ in time Δt; i.e., they have suffered an average retardation (or negative acceleration) in a Y-direction of

$$p_y = \left\{ \begin{matrix} \text{neg. accelera-} \\ \text{tion} \parallel \text{to } Y \end{matrix} \right\} = \frac{(c - v) \sin \alpha}{\Delta t} \quad . \quad . \quad . \quad . \quad (1)$$

Hence the resistance in direction of Y (i.e., the equal and opposite of P in figure) must be

$$P_Y = \text{mass} \times Y\text{-accel.} = \frac{\Delta M}{\Delta t}(c - v) \sin \alpha; \quad . \quad . \quad (2)$$

and therefore, since $\frac{\Delta M}{\Delta t} = M = \frac{Q\gamma}{g}$ = mass of liquid passing *over the plate* per unit of time (not that issuing from nozzle), we have

$$\left.\begin{array}{c}\textit{Impulse or pres-}\\ \textit{sure on plate}\end{array}\right\} = P = \frac{Q\gamma}{g}(c-v)\sin\alpha = \frac{F\gamma}{g}(c-v)^2\sin\alpha, \quad (3)$$

in which $F =$ sectional area of jet before meeting plate.

[N.B. Since eq. (3) can also be written $P = Mc\sin\alpha - Mv\sin\alpha$, and $Mc\sin\alpha$ may be called the Y-momentum before contact, while $Mv\sin\alpha$ is the Y-momentum after contact (of the mass passing over plate per unit of time), this method may be said to be founded on the *principle of momentum* which is nothing more than the relation that the accelerating force in any direction = mass × acceleration in that direction; e.g., $P_x = Mp_x$; $P_y = Mp_y$; see § 74.]

If we resolve P, Fig. 637, into two components, one, P', $\parallel$ to the direction of motion (v and c), and the other, P'', $\urcorner$ to the same, we have

$$P' = P\sin\alpha = \frac{Q\gamma}{g}(c-v)\sin^2\alpha, \quad . \quad . \quad . \quad . \quad (4)$$

and

$$P'' = P\cos\alpha = \frac{Q\gamma}{g}(c-v)\sin\alpha\cos\alpha. \quad . \quad . \quad (5)$$

($Q = F(c-v) =$ volume passing over the *plate* per unit of time.) The force P'' does no work, while the former, P', does an amount of work $P'v$ per unit of time; i.e., exerts a *power* (one plate)

$$= L = P'v = \frac{Q\gamma}{g}(c-v)v\sin^2\alpha. \quad . \quad . \quad . \quad (6)$$

If, instead of a single plate, a *series of plates*, forming a regular succession, is employed, then, as in a previous paragraph, we may replace Q, $= F(c-v)$, by $Q' = Fc$, obtaining as the

$$\left.\begin{array}{c}\textit{Power exerted by jet}\\ \textit{on series of plates}\end{array}\right\} = L' = \frac{Fc\gamma}{g}(c-v)v\sin^2\alpha. \quad . \quad (7)$$

For $v = \frac{c}{2}$ and $\alpha = 90°$ we have

$$L'_{\text{max.}} = \frac{1}{2}\frac{Fc\gamma}{g}\frac{c^2}{2} = \frac{1}{2}\frac{M'c^2}{2} \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

$=$ only half the kinetic energy (per time-unit) of the jet.

569. Rigid Plates Moving in a Fluid, Totally Submerged. Fluid Moving against a Fixed Plate. Impulse and Resistance.— If a thin flat rigid plate have a motion of uniform *translation* with velocity $= v$ through a fluid which completely surrounds it, Fig. 638, a resistance is encountered (which must be overcome by an equal and opposite force, not shown in figure. to preserve the uniform motion) consisting of a normal component N, ⏋ to plate, and a (small) tangential component, or skin-friction, T, || to plate. Unless the angle α, between the surface of plate and the direction of motion $O \ldots v$, is very small, i.e. unless the plate is moving nearly edgewise through the fluid, N is usually much greater than T. The skin-resistance between a solid and a fluid has already been spoken of in § 510.

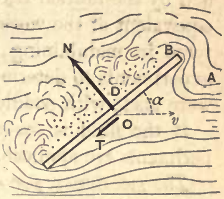


FIG. 638.

When the plate and fluid are at rest the pressures on both sides are normal and balance each other, being ordinary static fluid pressures. When motion is in progress, however, the normal pressures on the front surface are increased by the components, normal to plate, of the centrifugal forces of the curved filaments (such as AB) or "stream-lines," while on the back surface, D, the fluid does not close in fast enough to produce a pressure equal to that (even) of rest. In fact, if the motion is sufficiently rapid, and the fluid is inelastic (a liquid), a *vacuum may be maintained behind the plate*, in which case there is evidently no pressure on that side of the plate.

Whatever pressure exists on the back acts, of course, to diminish the resultant resistance. The water on turning the sharp corners of the plate is broken up into eddies forming a "wake" behind. From the accompaniment of these eddies, the resistance in this case (at least the component N normal to plate) is said to be due to "*eddy-making*;" though logically we should say, rather, that the body does not derive the assistance (or negative resistance) from behind which it would obtain if eddies were not formed; i.e., if the fluid could close in behind in smooth curved stream-lines symmetrical with those in front.

The *heat* corresponding to the change of temperature produced in the portion of fluid acted on, by the skin-friction and by the mutual friction of the particles in the eddies, is the equivalent of the work done (or energy spent) by the motive force in maintaining the uniform motion (§ 149). (Joule's experiments to determine the Mechanical Equivalent of Heat were made with paddles moving in water.)

If the fluid is *sea-water*, the results of Col. Beaufoy's experiments are applicable, viz.:

The resistance, per square foot of area, sustained by a submerged plate moving normally to itself [i.e., $\alpha = 90°$] *in sea-water with a velocity of* $v = 10$ *ft. per second is* 112 *lbs.* He also asserts that *for other velocities the resistance varies as the square of the velocity.* This latter fact we would be led to suspect from the results obtained in § 568 for the impulse of jets; also in § 565 [see eq. (6)]. Also, that when the plate moved obliquely to its normal (as in Fig. 638) *the resistance was nearly equal to (the resistance, at same velocity, when* $\alpha = 90°$) × (*the sine of the angle* α)*; also, that the depth of submersion had no influence on the resistance.*

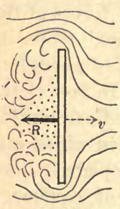


FIG. 639.

Confining our attention to a plate *moving normally to itself*, Fig. 639, let $F =$ area of plate, $\gamma =$ heaviness (§ 409) of the fluid, $v =$ the uniform velocity of plate, and $g =$ the acceleration of gravity ($= 32.2$ for the foot and second; $= 9.81$ for the metre and second). Then from the analogy of eq. (6), § 565, where velocity c of the jet against a stationary plate corresponds to the velocity v of the plate in the present case moving through a fluid at rest, we may write

$$\left.\begin{matrix}\textit{Resistance of fluid}\\ \textit{to moving plate}\end{matrix}\right\} = R = \zeta F\gamma\frac{v^2}{2g}\ldots\left\{\begin{matrix}v\text{ normal}\\ \text{to plate}\end{matrix}\right\}\ldots(1)$$

And similarly for the *impulse of an indefinite stream of fluid against a fixed plate* (⏋ to velocity of stream), v being the velocity of the current,

$$\left.\begin{matrix}\textit{Impulse of current}\\ \textit{upon fixed plate}\end{matrix}\right\} = P = \zeta' F\gamma\frac{v^2}{2g}\ldots\left\{\begin{matrix}v\text{ normal}\\ \text{to plate}\end{matrix}\right\}\ldots(2)$$

The $2g$ is introduced simply for convenience; since, having v given, we may easily find $v^2 \div 2g$ from a table of velocity-heads; and also (a ground of greater importance) since the coefficients ζ and ζ' which depend on experiment are evidently *abstract numbers* in the present form of these equations (for R and P are forces, and $F\gamma v^2 \div 2g$ is the weight (force) of an ideal prism of fluid; hence ζ and ζ' must be abstract numbers.)

From Col. Beaufoy's experiments (see above), we have for *sea-water* [ft., lb., sec.], putting $R = 112$ lbs., $F = 1$ sq. ft., $\gamma = 64$ lbs. per cub. ft., and $v = 10$ ft. per second,

$$\zeta = \frac{2 \times 32.2 \times 112}{1.0 \times 64 \times 10^2} = 1.13.$$

Hence in eq. (1) for sea-water, we may put $\zeta = 1.13$ (with $\gamma = 64$ lbs. per cub. ft.).

From the experiments of Dubuat and Thibault, Weisbach computes that for the plate of Fig. 639, moving through either water or air, $\zeta = 1.25$ for eq. (1), in which the γ for air must be computed from § 437; while for the impulse of water or air on fixed plates he obtains $\zeta' = 1.86$ for use in eq. (2). It is hardly reasonable to suppose that ζ and ζ' should not be identical in value, and Prof. Unwin thinks that the difference in the numbers just given must be due to errors of experiment. The latter value, $\zeta' = 1.86$, agrees well with equation (6) below. For great velocities ζ and ζ' are greater for air than for water, since air, being compressible, is of greater heaviness in front of the plate than would be computed for the given temperature and barometric height for use in eqs. (1) and (2)

The experiments of Borda in 1763 led to the formula

$$P = [0.0031 + 0.00035c]Sv^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

for the total pressure upon a plate moving through the air in a direction ⅂ to its own surface. P is the pressure in pounds, c the length of the contour of the plate in feet, and S its surface in square feet, while v is the velocity in miles per hour. Adopting the same form of formula, Hagen found, from experiments in 1873, the relation

$$P = [0.002894 + 0.00014c]Sv^2 \quad . \quad . \quad . \quad . \quad (4)$$

for the same case of fluid resistance.

Hagen's experiments were conducted with great care, but like Borda's were made with a "whirling machine," in which the plate was caused to revolve in a horizontal circle of only 7 or 8 feet radius at the end of a horizontal bar rotating about a vertical axis. Hagen's plates ranged from 4 to 40 sq. in. in area, and the velocities from 1 to 4 miles per hour.

The last result was quite closely confirmed by Mr. H. Allen Hazen at Washington in November 1886, the experiments being made with a whirling machine and plates of from 16 to 576 sq. in. area. (See the *American Journal of Science*, Oct. 1887, p. 245.)

In Thibault's experiments plates of areas 1.16 and 1.531 sq. ft. were exposed to direct wind-pressure, giving the formula

$$P = 0.00475Sv^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Recent experiments in France (see *R. R. and Eng. Journal*, Feb. '87), where flat boards were hung from the side of a railway train run at different velocities, gave the formula

$$P = 0.00535Sv^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The highest velocity was 44 miles per hour. The magnitude of the area did not seemingly affect the relation given. More extended and elaborate experiments are needed in this field, those involving a motion of translation being considered the better, rather than with whirling machines, in which "centrifugal action" must have a disturbing influence.

The notation and units for eqs. (4), (5), and (6) are the same as those given for (3).

It may be of interest to note that if equation (3) of § 568 be considered applicable to this case of the pressure of an unlimited stream of fluid against a plate placed at right-angles to the current, with F put equal to the area of the plate, we obtain, after reduction to the units prescribed above for the preceding equations and putting $\alpha = 90°$,

$$P = 0.0053 S v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The value $\gamma = 0.0807$ lbs. per cub. ft. has been used in the substitution, corresponding to a temperature of freezing and a barometric height of 30 inches. At higher temperatures, of course, γ would be less, unless with very high barometer.

569a. Example.—Supposing each blade of the paddle-wheel of a steamer to have an area of 6 sq. ft., and that when in the lowest position its velocity [relatively to the water, not to the vessel] is 5 ft. per second; what resistance is it overcoming in salt water?

From eq. (1) of § 569, with $\zeta = 1.13$ and $\gamma = 64$ lbs. per cubic foot, we have (ft., lb., sec.)

$$R = \frac{1.13 \times 6 \times 64 \times 25}{2 \times 32.2} = 169.4 \text{ lbs.}$$

If on the average there may be considered to be three paddles always overcoming this resistance on each side of the boat, then the work lost (work of "*slip*") in overcoming these resistances per second (i.e., *power* lost) is

$$L_1 = [6 \times 169.4] \text{ lbs.} \times 5 \text{ ft. per sec.} = 5082 \text{ ft.-lbs. per sec.}$$

or 9.24 Horse Power (since $5082 \div 550 = 9.24$).

If, further, the velocity of the boat is *uniform* and = 20 ft. per sec., the resistance of the water to the progress of the boat at this speed being 6 × 169.4, i.e. 1016.4 lbs., the power expended in actual propulsion is

$$L_2 = 1016.4 \times 20 = 20328 \text{ ft.-lbs. per sec.}$$

Hence the power expended in both ways (usefully in propulsion, uselessly in "slip") is

$$L_2 + L_1 = 25410 \text{ ft.-lbs. per sec.} = 46.2 \text{ H. P.}$$

Of this, 9.24 H. P., or about 20 per cent, is lost in "slip."

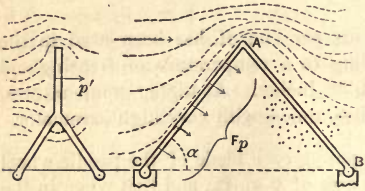


FIG. 640.

570. Wind-pressure on the surface of a roof inclined at an angle = α with the horizontal, i.e., with the direction of the wind, is usually estimated according to the empirical formula (Hutton's)

$$p = p' [\sin \alpha]^{[1.84 \cos \alpha - 1]}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which p' = pressure of wind per unit area against a *vertical* surface (⌉ to wind), and p = that against the inclined plane (*and normal to it*) at the same velocity. For a value of $p' = 40$ lbs. per square foot (as a maximum), we have the following values for p, computed from (1):

For α =	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°
p =(lbs. sq. ft.)	5.2	9.6	14	18.3	22.5	26.5	30.1	33.4	36.1	38.1	39.6	40.

Duchemin's formula for the normal pressure per unit-area is

$$p = p' \cdot \frac{2 \sin^2 \alpha}{1 + \sin^2 \alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with the same notation as above. Some experimenters in London tested this latter formula by measuring the pressure on a metal plate supported in front of the blast-pipe of a blowing engine; the results were as follows:

$\alpha =$	15°	20°	60°	90°
p by experiment = (in lbs. per sq. ft.)	1.65	2.05	3.01	3.31
By Duchemin's formula $p =$	1.60	2.02	3.27	3.31

The scale of the Smithsonian Institution at Washington for the estimation and description of the velocity and pressure of the wind is as follows:

Grade.	Velocity in miles per hour.	Pressure per sq. foot in lbs.	Name.
0	0	0.00	Calm.
1	2	0.02	Very light breeze.
2	4	0.08	Gentle breeze.
3	12	0.75	Fresh wind.
4	25	3.00	Strong wind.
5	35	6	High wind.
6	45	10	Gale.
7	60	18	Strong gale.
8	75		Violent gale.
9	90		Hurricane.
10	100		Most violent hurricane.

571. Mechanics of the Sail-boat.—The action of the wind on a sail will be understood from the following. Let Fig. 641 represent the boat in horizontal projection and OS the sail, O being the mast. For simplicity we consider the sail to be a plane and to remain vertical. At this instant the boat is moving in the direction MB of its fore-and-aft line with a velocity $= c$, the wind having a velocity of the direction and magnitude represented by w (purposely taken at an angle $< 90°$ with the direction of motion of the boat). We are now to inquire the nature of the action of the wind on the boat, and whether

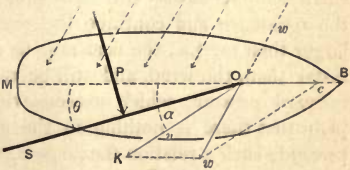


Fig. 641.

in the present position its tendency is to accelerate, or retard, the motion of the boat. If we form a parallelogram of which w is the diagonal and c one side, then the other side OK, making some angle α with BM, will be the velocity v of the wind *relatively to the boat* (and sail), and upon this (and not upon w) depends the action on the sail. The sail, being so placed that the angle θ is smaller than α, will experience pressure from the wind; that is, from the impact of the particles of air which strike the surface and glance along it. This pressure, P, is normal to the sail (considered smooth), and evidently, for the position of the parts in the figure, the component of P along MB points in the same direction as c, and hence if that component is greater than the water-resistance to the boat at this velocity, c will be accelerated; if less, c will be retarded. Any change in c, of course, gives a different form to the parallelogram of velocities, and thus the relative velocity v and the pressure P, for a given position of the sail, will both change. [The component of P ⏋ to MB tends, of course, to cause the boat to move laterally, but the great resistance to such movement at even a very slight lateral velocity will make the resulting motion insignificant.]

As c increases, α diminishes, for a given amount and position of w; and the sail must be drawn nearer to the line MB, i.e. θ must be made to decrease, to derive a wind-pressure having a *forward* fore-and-aft component; and that component becomes smaller and smaller. But if the craft is an ice-boat, this small component may still be of sufficient magnitude to exceed the resistance and continue the acceleration of c until c is larger than w; i.e., the boat may be caused to go as fast as, or faster than, the wind, and still be receiving from the latter a forward pressure which exceeds the resistance. And it is plain that there is nothing in the geometry of the figure to preclude such a relation (i.e., $c > w$, with $\theta < \alpha$ and > 0).

572. Resistance of Still Water to Moving Bodies, Completely Immersed.—This resistance depends on the shape, position, and velocity of the moving body, and also upon the roughness of its surface. If it is pointed at both ends (Fig. 642) with its axis parallel to the velocity, v, of its *uniform* motion, the stream-lines on closing together smoothly at the hinder extremity, or stern, B, exert normal pressures against the surface of the portion $CD...B$ whose longitudinal components approximately balance the corresponding components of the normal pressures on $CD \ldots A$; so that the resistance R, which must be overcome to maintain the uniform velocity v, is *mainly due to the* "*skin-friction*" alone, distributed along the external surface of the body; the resultant of these resistances is a force R acting in the line AB of symmetry (supposing the body symmetrical about the direction of motion).

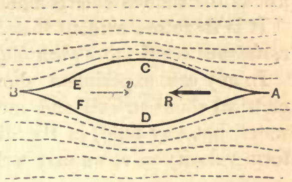


FIG 642

If, however, Fig. 643, the stern, $E..B..F$ is too bluff, eddies are formed round the corners E and F, and the pressure on the surface $E \ldots F$ is much less than in Fig. 642; i.e., the water pressure from behind is less than the backward (longitudinal) pressures from in front, and thus the resultant resistance R is due partly to skin-friction and partly to "eddy-making" (§ 569).

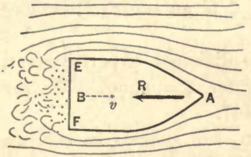


FIG. 643.

[NOTE.—The diminished pressure on EF is analogous to the loss of pressure of water (flowing in a pipe) after passing a narrow section the enlargement from which to the original section is sudden. E.g., Fig. 644, supposing the velocity v and pressure p (per unit-area) to be the same respectively at A and A', in the two pipes shown, with diameter $AL = WK = A'L' = W'K'$; then the pressure at M is equal to that at A (disregarding skin-friction), whereas that at

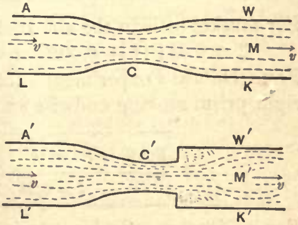


FIG. 644.

M' is considerably *less* than that at A' on account of the head lost in the sudden enlargement. (See also Fig. 575.)]

It is therefore evident that *bluffness of stern increases the resistance much more than bluffness of bow.*

In any case experiment shows that for a given body symmetrical about an axis and moving through a fluid (not only water, but any fluid) in the direction of its axis with a uniform velocity $= v$, we may write approximately the resistance

$$R = (\textit{resistance at vel. } v) = \zeta F \gamma \frac{v^2}{2g}. \quad . \quad . \quad . \quad (1)$$

As in preceding paragraphs, $F =$ area of the *greatest* section, ┐ to axis, of the external *surface* of body (not of the substance); i.e., the sectional area of the circumscribing cylinder (cylinder in the most general sense) with elements parallel to the axis of the body. $\gamma =$ the heaviness (§ 409) of the fluid, and $v =$ velocity of motion; while ζ is an *abstract number* dependent on experiment.

According to Weisbach, who cites different experimenters, we can put for *spheres*, moving in water, $\zeta =$ about 0.55; for cannon-balls moving in water, $\zeta = .467$.

According to Robins and Hutton, for *spheres* in *air*, we have

For v in mets. per sec.	1	5	25	100	200	300	400	500 metres per sec.
$\zeta =$	.59	.63	.67	.71	.77	.88	.99	1.04

For musket-balls in the air, Piobert found

$$\zeta = 0.451\ (1 + 0.0023 \times \text{veloc. in metres per sec.}).$$

From Dubuat's experiments, for the resistance of water to a right prism moving endwise and of length $= l$,

For $(l : \sqrt{F}) =$	0	1	2	3
$\zeta =$	1.25	1.26	1.31	1.33

For a circular cylinder moving perpendicularly to its axis Borda claimed that ζ is one-half as much as for the circumscribing right parallelopiped moving with four faces parallel to direction of motion.

EXAMPLE.—The resistance of the air at a temperature of freezing and tension of one atmosphere to a musket-ball $\frac{1}{2}$ inch in diameter when moving with a velocity of 328 ft. per sec. is thus determined by Piobert's formula, above:

$$\zeta = 0.451(1 + .0023 \times 100) = 0.554;$$

hence, from eq. (1),

$$R = 0.554 \times \frac{\pi}{4}\left(\frac{\frac{1}{2}}{12}\right)^2 \times .0807 \times \frac{(328)^2}{64.4} = 0.1018 \text{ lbs.}$$

572a. Deviation of a Spinning Ball from a Vertical Plane in Still Air.—It is a well-known fact in base-ball playing that if a rapid spinning motion is given to the ball about a vertical axis as well as a forward motion of translation, its path will not remain in its initial vertical plane, but curve out of that plane toward the side on which the absolute velocity of an external point of the ball's surface is least. Thus, if the ball is thrown from North to South, with a spin of such character as to appear "*clock-wise*" seen from above, the ball will curve *toward the West*, out of the vertical plane in which it started.

This could not occur if the surface of the ball were perfectly smooth (there being also no adhesion between that surface and the air particles), and is due to the fact that the cushion of compressed air which the ball piles up in front during its progress, and which would occupy a symmetrical position with respect to the direction of motion of the centre of the ball if there were no motion of rotation of the kind indicated, is now piled up somewhat on the East of the centre (in example above), creating constantly more obstruction on that side than on the right; the cause of this is that the absolute velocity of the surface-points, at the same level as the centre of ball, is greatest, and the friction greatest, at the instant when they are passing through their extreme Easterly positions; since then that velocity is the *sum* of the linear velocity of translation and that of rotation; whereas, in the position diametrically opposite, on the West side, the absolute velocity is the *difference*; hence the greater accumulation of compressed air on the left (in the case above imagined, ball thrown from North to South, etc.).

573. Robinson's Cup-anemometer.—This instrument, named after Dr. T. R. Robinson of Armagh, Ireland, consists of four hemispherical cups set at equal intervals in a circle, all facing in the same direction round the circle, and so mounted on a light but rigid framework as to be capable of rotating with but little friction about a vertical axis. When in a current of air (or other fluid) the apparatus begins to rotate with an accelerated velocity on account of the pressure against the open mouth of a cup on one side being greater than the resistance met by the back of the cup diametrically opposite. Very soon, however, the motion becomes practically uniform, the cup-centre having a constant linear velocity v'' the ratio of which to the velocity, v', of the wind at the same instant must be found in some way, in order to deduce the value of the latter from the observed amount of the former in the practical use of the instrument. After sixteen experiments made by Dr. Robinson on stationary cups exposed to winds of varying intensities, from a gentle breeze to a hard gale, the conclusion was reached by him that with a given wind-velocity the total pressure on a cup with concave surface presented to the wind was very nearly four times as great as that exerted when the convex side was presented, whatever the velocity (see vol. XXII of *Transac. Irish Royal Acad.*, *Part I*, p. 163).

Assuming this ratio to be exactly 4.0 and neglecting axle-friction, we have the data for obtaining an approximate value of m, the ratio of v' to the observed v'', when the instrument is in use. The influence of the wind on those cups the planes of whose mouths are for the instant $\parallel$ to its direction will also be neglected.

If, then, Fig. 645, we write the *impulse* on a cup when the hollow is presented to the wind [§ 572, eq. (1)]

$$P_h = \zeta_h F\gamma \frac{v_1^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the *resistance* when the convex side is presented

$$P_c = \zeta_c F\gamma \frac{v_2^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we may also put

$$\zeta_h = 4\zeta_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In (1) and (2) v_1 and v_2 are relative velocities.

Regarding only the two cups A and B, whose centres at a definite instant are moving in lines parallel to the direction of the wind, it is evident that the motion of the cups does not become uniform until the relative velocity $v' - v''$ of the wind and cup A (retreating before the wind) has become so small, and the relative velocity $v' + v''$ with which B advances to meet the air-particles has become so great, that the impulse of the wind on A *equals* the resistance encountered by B; i.e., these forces, P_h and P_c, must be equal, having equal lever-arms about the axis. Hence, for uniform rotary motion,

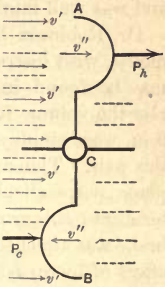


FIG. 645.

$$\zeta_h F\gamma \frac{(v'-v'')^2}{2g} = \zeta_c F\gamma \frac{(v'+v'')^2}{2g}; \quad . \quad . \quad . \quad . \quad (4)$$

i.e. [see eq. (3)],

$$4\left[\frac{v'}{v''} - 1\right]^2 = \left[\frac{v'}{v''} + 1\right]^2; \quad \text{or,} \quad 4(m-1)^2 = (m+1)^2. \quad . \quad (5)$$

Solving the quadratic for m, we obtain

$$m = 3.00. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is, the velocity of the wind is about three times that of the cup-centre.

574. Experiments with Robinson's Cup-anemometer.—The ratio 3.00 just obtained is the one in common use in connection with this instrument in America. Experiments by Mr.

Hazen at Washington in 1886 (*Am. Jour. Science*, Oct. '87, p. 248) were made on a special type devised by Lieut. Gibbon. The anemometer was mounted on a whirling machine at the end of a 16-ft. horizontal arm, and values for *m* obtained, with velocities up to 12 miles per hour, from 2.84 to 3.06; average 2.94. The cups were 4 in. in diameter and the distance of their centres from the axis 6.72 in., these dimensions being those usually adopted in America. This instrument was nearly new and was well lubricated.

Dr. Robinson himself made an extensive series of experiments, with instruments of various sizes, of which an account may be found in the *Philos. Transac.* for 1878, p. 797 (see also the volume for 1880, p. 1055). Cups of 4 in. and also of 9 in. were employed, placed first at 24 and then at 12 in. from the axis. The cup-centres revolved in a (moving) vertical plane perpendicular to the horizontal arm of a whirling-machine; this arm, however, was only 9 ft. long. A friction-brake was attached to the axis of the instrument for testing the effect of increased friction on the value of *m*. At high speeds of 30 to 40 miles per hour (i.e., the speed of the centre of the instrument in its horizontal circle, representing an equal speed of wind for an instrument in actual use with axis stationary) the effect of friction was relatively less than at low velocities. That is, at high speeds with considerable friction the value of *m* was nearly the same as with little friction at low speeds. With the large 9 in. cups at a distance of either 24 or 12 in. from the axis the value of *m* at 30 miles per hour ranged generally from 2.3 to 2.6, with little or much friction; while with the minimum friction *m* rose slowly to about 2.9 as the velocity diminished to 10 miles per hour. At 5 miles per hour with minimum friction *m* was 3.5 for the 24 in. instrument and about 5.0 for the 12 in. The effect of considerable friction at low speeds was to increase *m*, making it as high as 8 or 10 in some cases. With the 4 in.-cups no value was obtained for *m* less than 3.3. On the whole, Dr. Robinson concluded that *m* is more likely to have a constant value at all velocities the larger the cups, the longer the arms, and the less the friction, of the anemometer. But few straight-line experiments have been made with the cup-anemometer, the most noteworthy being mentioned on p. 308 of the *Engineering News* for October 1887. The instrument was placed on the front of the locomotive of a train running between Baltimore and Washington on a calm day. The actual distance is 40 miles between the two cities, while from the indications of the anemometer, assuming $m = 3.00$, it would have been in one trip 46 miles and in another 47. The velocity of the train was 20 miles per hour in one case and 40 in the other.

575 Other Anemometers.—Both Biram's and Castello's anemometers consist of a wheel furnished with radiating vanes set obliquely to the axis of the wheel, forming a small "windmill," somewhat resembling the current-meter for water shown in Fig. 604; having six or eight blades, however. The wheel revolves with but little friction, and is held in the current of air with its axis parallel to the direction of the latter, and very quickly assumes a steady motion of rotation. The number of revolutions in an observed time is read from a dial. The instruments must be rated by experiment, and are used chiefly in measuring the velocity of the currents of air in the galleries of mines, of draughts of air in flues and ventilating shafts, etc.

To quote from vol. v of the Report of the Geological Survey of Ohio, p. 370: "Approximate measurements (of the velocity of air) are made by miners by flashing gunpowder, and noting with a watch the speed with which the smoke moves along the air-way of the mine. A lighted lamp is sometimes used, the miner moving along the air-gallery, and keeping the light in a perfectly perpendicular position, noting the time required to pass to a given point."

Another kind makes use of the principle of Pitot's Tube (p. 751), and consists of a U-tube partially filled with water, one end of the tube being vertical and open, while the other turns horizontally, and is enlarged into a wide funnel, whose mouth receives the impulse of the current of air; the difference of level of the water in the two parts of the U is a measure of the velocity.

576. Resistance of Ships.—We shall first suppose the ship *to be towed* at a uniform speed; i.e., to be without means of self-propulsion (under water). This being the case, it is found that at moderate velocities (under six miles per hour), the ship being of "*fair*" form (i.e., the hull tapering both at bow and stern, under water) the resistance in still water is almost wholly due to *skin-friction*, "eddy-making" (see § 569) being done away with largely by avoiding a bluff stern.

When the velocity is greater than about six miles an hour the resistance is much larger than would be accounted for by skin-friction alone, and is found to be connected with the surface-disturbance or waves produced by the motion of the hull in (originally) still water. The recent experiments of Mr. Froude and his son at Torquay, England, with models, in a tank 300 feet long, have led to important rules (see Mr. White's *Naval Architecture* and "Hydromechanics" in the *Ency. Britann.*) of so proportioning not only the total length of a ship of given displacement, but the length of the *entrance* (forward tapering part of hull) and length of *run* (hinder tapering part of hull), as to secure a minimum "*wave-making resistance*," as this source of resistance is called.

To quote from Mr. White (p. 460 of his *Naval Architecture*, London, 1882): "Summing up the foregoing remarks, it appears:

"(1) That *frictional resistance*, depending upon the area of the immersed surface of a ship, its degree of roughness, its length, and (about) the square of its speed, is not sensibly affected by the forms and proportions of ships; unless there be some unwonted singularity of form, or want of fairness. For *moderate speeds* this element of resistance is by far the most important; for high speeds it also occupies an important position—from 50 to 60 per cent of the whole resistance, probably, in a very large number of classes, when the bottoms are *clean;* and a larger percentage when the bottoms become foul.

"(2) That *eddy-making resistance* is usually small, except in special cases, and amounts to 8 or 10 per cent of the frictional resistance. A defective form of stern causes largely increased eddy-making.

"(3) That *wave-making resistance* is the element of the total resistance which is most influenced by the forms and proportions of ships. Its ratio to the frictional resistance, as well as its absolute magnitude, depend on many circumstances; the most important being the forms and lengths of the entrance and run, in relation to the intended full speed of the ship. For every ship there is a limit of speed beyond which each small increase in speed is attended by a disproportionate increase in resistance; and this limit is fixed by the lengths of the entrance and run—the 'wave-making features' of a ship.

"The sum of these three elements constitutes the total resistance offered by the water to the motion of a ship towed through it, or propelled by sails; in a steamship there is an '*augment*' of resistance due to the action of the propellers."

In the case of a steamship driven by a screw propeller, this *augment* to the resistance varies from 20 to 45 per cent of the "tow-rope resistance," on account of the presence and action of the propeller itself; since its action relieves the stern of some of the *forward* hydrostatic pressure of the water closing in around it. Still, if the screw is placed far back of the stern, the augment is very much diminished; but such a position involves risks of various kinds and is rarely adopted.

We may compute approximately the resistance of the water to a ship propelled by steam at a uniform velocity v, in the following manner: Let L denote the power developed in the engine cylinder; whence, allowing 10 per cent of L for engine friction, and 15 per cent for "work of slip" of the propeller-blade, we have remaining $0.75L$, as expended in overcoming the resistance R through a distance $= v$ each unit of time; i.e.,

$$(approx.) \qquad 0.75L = Rv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Example.—If 3000 indicated H. P. (§ 132) is exerted by the engines of a steamer at a uniform speed of 15 miles per hour

(= 22 ft. per sec.), we have (with above allowances for slip and engine friction) [foot-lb.-sec.]

$$\tfrac{3}{4} \times 3000 \times 550 = R \times 22 ; \quad \therefore \quad R = 56250 \text{ lbs.}$$

Further, since R varies (roughly) as the *square* of the velocity, and can therefore be written $R = (\text{Const.}) \times v^2$, we have from (1)

$$L = a\ constant \times v^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as a roughly approximate relation between the speed and the power necessary to maintain it uniformly. In view of eq. (3) involving the *cube* of the velocity as it does, we can understand why a large increase of power is necessary to secure a proportionally small increase of speed.

577. "Transporting Power," or Scouring Action, of a Current.—The capacity or power of a current of water in an open channel to carry along with it loose particles, sand, gravel, pebbles, etc., lying upon its bed was investigated experimentally by Dubuat about a century ago, though on a rather small scale. His results are as follows:

The velocity of current must be at least

0.25	ft.	per sec.,	to transport	silt;
0.50	"	"	"	loam;
1.00	"	"	"	sand;
2.00	"	"	"	gravel;
3.5	"	"	"	pebbles 1 in. in diam.;
4.0	"	"	"	broken stone;
5.0	"	"	"	chalk, soft shale.

However, more modern writers call attention to the fact that in some instances beds of sand are left undisturbed by currents of greater velocity than that above indicated for sand, and explain this fact on the theory that the water-particles may not move parallel to the bed, but in cycloids, approximately, like the points in the rim of a rolling wheel, so as to have little or no scouring action on the bed in those cases.

In case the particles move in filaments or stream-lines parallel to the axis of the stream the statement is sometimes made that *the "transporting power" varies as the sixth power* *of the velocity of the current*, by which is meant, more definitely, the following: Fig. 646. Conceive a row of cubes (or other solids geometrically *similar* to each other) of many sizes, all of the same heaviness (§ 7), and similarly situated, to be placed on the horizontal bottom of a trough and there exposed to a current of water, being completely immersed. Suppose the coefficient of friction between the cubes and the trough-bottom to be the same for all. Then, as the current is given greater and greater velocity v, the impulse P_m (corresponding to a particular velocity v_m) against some one, m, of the cubes, will be just sufficient to move it, and at some higher velocity v_n the impulse P_n against some larger cube, n, will be just sufficient to move it, in turn. We are to prove that $P_m : P_n :: v_m^6 : v_n^6$.

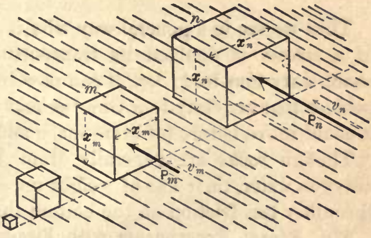


FIG. 646.

Since, when a cube barely begins to move, the impulse is equal to the friction on its base, and the frictions under the cubes (when motion is impending) are proportional to their volumes (see above), we have therefore

$$\frac{P_m}{P_n} = \frac{x_m^3}{x_n^3} \quad \ldots \ldots \ldots \quad (1)$$

Also, the impulses on the cubes, whatever the velocity, are proportional to the face areas and to the squares of the velocities (nearly; see § 572); hence

$$\frac{P_m}{P_n} = \frac{v_m^2 x_m^2}{v_n^2 x_n^2} \quad \ldots \ldots \ldots \quad (2)$$

From (1) and (2) we have

$$\frac{v_m^2}{v_n^2} = \frac{x_m}{x_n}; \text{ i.e., } \frac{x_m^2}{x_n^2} = \frac{v_m^4}{v_n^4}; \quad \ldots \quad (3)$$

while from (3) and (2) we have, finally,

$$P_m : P_n :: v_m^6 : v_n^6. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus we see in a general way why it is that if the velocity of a stream is doubled its transporting power is increased about sixty-four-fold; i.e., it can now impel along the bottom pebbles that are sixty-four times as heavy as the heaviest which it could move before (of same shape and heaviness).

Though rocks are generally from two to three times as heavy as water, their loss of weight under water causes them to encounter less friction on the bottom than if not immersed.

578. Recent Experiments with Fire-hose, Nozzles, etc. (Addendum to § 520.)—The very full and careful investigations of Mr. J. R. Freeman, hydraulic engineer of Boston, Mass., in this line (see Transac. A. S. C. E., Nov. 1889) furnish the following results: By taking piezometer readings at the ends of a portion of fire-hose conducting a steady flow of water, the values of loss of pressure due to fluid friction per 100 feet of length could be computed; a careful measurement being also made of the diameter of the hose and of the volume of water transmitted in an observed time. The table here given presents results applicable to hose of exactly 2.5 inches diameter, for a delivery of water at the rate of 240 gallons per minute (that is, for a velocity in the hose of 15.68 ft. per sec.). (The value of *f* has been computed by the writer.)

Sample.	Description.	Loss of pressure, per 100 ft. of length, in lbs. per sq. in.	Coefficient *f*. (See § 520.)	Velocity of water in hose.
L	Unlined linen hose	33.2	0.01045	15.68
K	Woven cotton, rubber-lined, "Mill Hose".....	25.5	0.00802	15.68
I	K. cotton, rubber-lined, hose.	19.4	0.00610	15.68
E	Ditto, but interstices between threads well filled............	16.0	0.00503	15.68
C	Woven cotton, rubber-lined, hose. So well filled with the rubber that the inner surface remained smooth under pressure	14.1	0.00443	15.68

It was found that with other rates of flow the friction-head varied nearly as the square of the velocity. The great importance of a smooth interior of hose is well shown by this table.

A short section of each kind of hose was filled with liquid plaster *under pressure*. After the setting of the plaster the hose was removed and photographs taken of the cast, thus conveying a definite idea of the degree of roughness of interior of hose.

As to nozzles, it was found that the plain conical nozzle gave the best results, jets from the ring-nozzles being slightly inferior in range.

By means of a very delicate form of Pitot's tube, measurements were made of the velocity in different parts of the section of jets, near the nozzle, with the interesting result that in "about two-thirds the whole distance from centre to circumference the velocity remains the same as at centre," and that at $\frac{1}{8}$ inch from the wall of most of the orifices the velocity was only 5% less than at centre of jet. With a jet from a 5-foot length of brass tubing $1\frac{1}{8}$ inch in diameter and used as a nozzle the velocity fell off rapidly for filaments further from the centre; e.g., at half the distance from centre to circumference the velocity was 90% of that at the centre, and at the outside edge 60%. Most of the nozzles ranged from 1 in. to $1\frac{1}{4}$ in. in diameter of orifice.

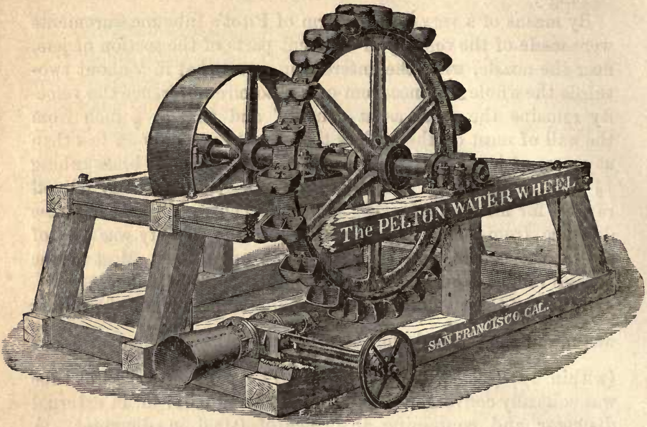
By using these velocity measurements to "gauge" the flow it was found that the relation $h' = \frac{c^2}{2g}$ was quite closely borne out (within 1%) (see eq. (7)″, p. 804). The point of the Pitot tube was conically convergent, its extremity being 0.017 in. in external diameter and containing an orifice of 0.006 in. diameter. A minute passage-way led from the orifice to a Bourdon gauge.

Based on his experiments, Mr. Freeman gives tables for the maximum vertical height, V, and also the maximum horizontal range, H, of "*good effective fire-streams*" delivered from smooth conical nozzles of various sizes and with different piezometer pressures p (in lbs. p. sq. in. above atmosphere) at the base of play-pipe, the gauge being at same level as nozzle. (The distances reached by the extreme drops are very much greater with the high pressures. V and H are in feet.)

The following is a brief synopsis of this table. d is the internal diameter of the extremity of nozzle. The maximum horizontal range was obtained at an angle of elevation of about 32°.

For $p =$	10		20		40		60		80		100	
	V	H	V	H	V	H	V	H	V	H	V	H
$d = \frac{3}{4}$ in.	17 ft.	19	33	29	60	44	72	54	79	62	83	68
$d = \frac{7}{8}$ "	18	21	34	33	62	49	77	61	85	70	90	76
$d = 1$ "	18	21	85	37	64	55	79	67	89	76	96	83
$d = 1\frac{1}{8}$ "	18	22	36	38	65	59	83	72	92	81	99	89
$d = 1\frac{1}{4}$ "	19	22	37	40	67	63	85	76	95	85	101	93
$d = 1\frac{3}{8}$ "	20	23	88	42	69	66	87	79	97	88	103	96

579. Addendum on the Pelton Water-wheel.—The annexed cut and additional details of the test alluded to on p. 810 are taken from circulars of the manufacturers, of San Francisco, Cal.



The water was measured over an iron weir ⅛″ thick and 3.042 feet long without end contraction.

The depth was measured by a Boyden hook gauge reading to .001″, and was .4146 foot. The quantity of water discharged was found to be 2.819 cubic feet per second—Fteley's formula. The head lost by friction in pipe was 1.8 feet, reducing the effective head to 384.7 feet.

The work done was measured by a Prony brake bearing vertically down upon a platform scale and which showed a weight of 200 pounds upon the scale-beam when the brake gear was suspended by a cord from a point immediately above the wheel-shaft. This made a constant minus correction of 200 pounds. The friction pulley had a face of 12″, and being kept wet by a jet of clear cold water, it developed very little heat and ran without much jumping. Thirteen tests were made showing very uniform results, the first four of which were as follows:

Tests.	Weight shown by scale.	Net weight (—200 lbs.)	Rev. wheel-shaft per min.	
1	665	$G = 465$	$u = 254\frac{1}{2}$	$Gu = 118342$
2	665	465	255	118575
3	660	460	256	117760
4	660	460	$256\frac{1}{2}$	117990
			Totals 1.022	472667
			Means $255\frac{1}{2}$	118167

The arm of the Prony brake was 4.775 feet from centre of wheel-shaft to point of contact on scale and hence described a circle with a circumference of 30 feet. The work done per minute was therefore $Gu(2\pi r) = (118167 \times 30)$ or 3,545,000 foot-pounds, equal to 107.4 horse-power. The theoretical power of the water was $(2.819 \times 60 \times 384.7 \times 62.4)$ or 4,060,253 foot-pounds. The useful effect was therefore 87.3 per cent.

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